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UNIVERSITY OF ALBERTA

GRADED GROUPS AND TOROIDAL GROUPS

by

ZHIYONG SHI

(C)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Graded Groups and Toroidal Groups submitted by ZHIYONG SHI in partial fulfillment of the degree of Doctor of Philosophy.

Supervisor Robert V. Moody

Arturo Pianzola

M. Shirvani

A. Weiss

R. J. Elliott

Date

DEDICATION

To my family

who are the representations

of my life

ABSTRACT

In this thesis, we study certain groups involved in Lie theory. Namely, these are groups graded by finite root systems, toroidal groups and toroidal Weyl groups.

A Steinberg group, associated with a finite root system and a ring, is an abstract group with certain generators and (Steinberg) relations. In Chpater 1, we introduce groups graded by finite root systems of type $A_\ell, \ell \geq 3, D_\ell, \ell \geq 4$, and $E_\ell, \ell = 6, 7, 8$, based on some properties of Steinberg groups. A ring can be constructed from such a group and then this group has Steinberg relations. Also we show that any covering of a group graded by a root system of rank at least 4 for A_ℓ and 5 for D_ℓ is also graded by this root system. In Chapter 2, by using integral bases of toroidal Lie algebras relative to simply laced finite root systems $\dot{\Delta}$, we construct toroidal groups, which generalizes affine Kac-Moody groups. Toroidal groups are homomorphic images of the Steinberg groups $St(\dot{\Delta}, \mathbb{C}[s^{\pm 1}, t^{\pm 1}])$. We study the toroidal groups via the vertex representations and show that in the case of $A_\ell, \ell \geq 4$, $St(\dot{\Delta}, \mathbb{C}[s^{\pm 1}, t^{\pm 1}])$ is a universal central extension of the toroidal group with the kernel $K_2(\mathbb{C})$. Also we show that finite Weyl groups, affine Weyl groups and toroidal Weyl groups are subquotient of the Steinberg groups. In Chpater 3, we discuss toroidal Weyl groups. It turns out a toroidal Weyl group is the semidirect product of the relative finite Weyl group and a discrete Heisenberg group.

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Chapter 0. Introduction

This thesis consists of three chapters and they are based on three relatively independent papers. For this reason , there are minor variations in notation from chapter to chapter.

In this thesis we only consider the simply laced finite root systems Δ, that is, Δ is of type $A_\ell, \ell \geq 1, D_\ell, \ell \geq 4$, or $E_\ell, \ell = 6, 7, 8$.

Root systems give gradings not only to Lie algebras ([BM]) but also to groups. For examples, Steinberg groups and Chevalley groups have *gradings* with respect to the group commutator. Recall that for a root system Δ and a (commutative) associative ring R, a Steinberg group ([HO], [Mi], [Ste], [St]), denoted by $St(\Delta, R)$, is defined by generators $\{\hat{x}_\alpha(r) : \alpha \in \Delta, r \in R\}$ and Steinberg relations. $St(\Delta, R)$ has a collection of "one parameter" subgroups $\mathcal{X}^\alpha, \alpha \in \Delta$, which generate $St(\Delta, R)$ and satisfy $\mathcal{X}^\alpha \cap \mathcal{X}^\beta = (1)$, if $\alpha \neq \beta$. For all $\alpha \in \Delta, u \in R^\times$ (the units group of R), let $\hat{n}_\alpha(u) := \hat{x}_\alpha(u)\hat{x}_\alpha(-u^{-1})\hat{x}_\alpha(u)$ and $\hat{h}_\alpha(u) := \hat{n}_\alpha(u)\hat{n}_\alpha(-1)$. Denote by $\hat{N}$ and $\hat{H}$ the subgroups of $St(\Delta, R)$ generated by all $\hat{n}_\alpha(u)$'s and $\hat{h}_\alpha(u)$'s, respectively. Then the defining relations imply

$$(\mathcal{X}^\alpha, \mathcal{X}^\beta) \subseteq \begin{cases} (1), & \text{if } \alpha + \beta \notin \Delta \cup \{0\} \\ \mathcal{X}^{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta, \end{cases}$$

where $(\cdot, \cdot)$ denotes the group commutator. Let K be a commutative subring of R with the same identity 1 of R. For example, if R is a K-algebra, then such a ring natually arises. The subring of R generated by the identity element 1 is also an example, which of course is isomorphic to one of the rings $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}, n \in \mathbb{Z}$. Let $\dot{\mathcal{X}}$ be the subgroup generated by $\hat{x}_\alpha(r), \alpha \in \Delta, r \in \mathsf{K}$ and $\dot{\mathcal{X}}^\alpha := \mathcal{X}^\alpha \cap \dot{\mathcal{X}}$. Suppose that $(\dot{G}, \dot{\phi})$ is a pair where $\dot{\phi}$ is a group homomorphism from $St(\Delta, \mathsf{K})$ onto $\dot{G}$ such that for some positive system Δ_+ of Δ, $\dot{\phi}|_{\mathcal{X}^+(\mathsf{K})}$ is one-to-one, where $\mathcal{X}^+(\mathsf{K}) = \langle \mathcal{X}^\alpha(\mathsf{K}) \mid \alpha \in \Delta_+ \rangle < St(\Delta, \mathsf{K})$. Let $\mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$ be the collection of all such pairs. Then $\dot{\mathcal{X}}^\alpha \subseteq \mathcal{X}^\alpha$ and $\dot{\mathcal{X}} \in \mathfrak{S}$ where the homomorphism is natural. Recall that the conjugation by $\hat{n}_\alpha(\pm 1)$ maps $\mathcal{X}^\beta$ onto $\mathcal{X}^{r_\alpha \beta}$, where r_α is the refelction in α. Based on these properties of $St(\Delta, R)$, but excluding the appearance of the ring

R and the generators and relations, we define Δ-graded groups. Then we show that these properties determine a Steinberg group-like structure for a Δ-graded group. Indeed, we will show that each subgroup G^α is coordinatized by a ring R, and then show that G is a homomorphic image of the Steinberg group $St(\Delta, R)$. This ring is associative, and furthermore is commutative if Δ is type of D_ℓ or E_ℓ. $St(\Delta, R)$ are Δ-graded. We also show that covering groups of Δ-graded groups are still Δ-graded and moreover if two groups have a same covering group and one of them is Δ-graded, so is the other. These are contents of Chapter 1.

In Chapter 2 , we shall study toroidal groups. Then it natually leads to the study of toroidal Weyl groups in Chapter 3. To agree with the notation there, we will now denote finite root systems by $\dot{\Delta}$ instead of Δ and Δ will mean toroidal root systems. For the purpose of notation, we describe Chapter 3 first.

In [MRY] and [EMY], Moody, Eswara Rao and Yokonuma studied the toroidal Lie algebras and their representations. A (n-)toroidal Lie algebra is a universal central extension of the Lie algebra $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \otimes \dot{\mathfrak{g}}$, where $\dot{\mathfrak{g}}$ is the finite dimensional simple Lie algebra over the complex field $\mathbb{C}$. The terminology "toroidal" is justified because the Lie algebra $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \otimes \dot{\mathfrak{g}}$ can be viewed as the Lie algebra of all polynomial maps from the complex torus $\mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$($n$ copies) to $\dot{\mathfrak{g}}$. When $n = 1$, the theory is nothing but the theory of the well known affine Lie algebras.

Loosely speaking, a toroidal Lie algebra is an n-fold affinization of $\dot{\mathfrak{g}}$. Generalizing the 1-fold affinization of root systems and Weyl groups, we may study n-fold root systems and n-fold Weyl groups. In Chapter 3, we reach this goal through the standard *geometric lattice* approach. Let $\dot{\Delta}$ be the finite root system of $\dot{\mathfrak{g}}$ (simply laced) and $\dot{Q}$ be the finite root lattice of $\dot{\Delta}$. Under the standard normalized bilinear form $(\cdot \mid \cdot)$, $\dot{\Delta}$ is the subset of vectors with square length 2 of $\dot{Q}$. Extend $\dot{Q}$ to a toroidal root lattice Q by adding n null roots such that Q inherits a symmetric bilinear form from $\dot{Q}$ with the null roots in its radical. We extend Q to a lattice on which $(\cdot \mid \cdot)$ is non-singular. The corresponding toroidal Weyl group is generated

by all reflections in *real roots* (these are vectors of Q with square length 2). We will characterize the toroidal Weyl group as a semidirect product of finite Weyl group and a discrete Heisenberg group with a center which is a free abelian group of rank $n(n-1)/2$. Toroidal Weyl groups are homomorphic images of some indefinite Kac-Moody Weyl groups and the kernels are given. We show how toriodal Weyl groups constructed from lattices naturally arise from toroidal Lie algebras.

For the finite root system $\dot{\Delta}$, $St(\dot{\Delta}, \cdot)$ is a functor from the category of (commutative) rings to the category of abelian groups. It can be generalized to the functor $St({}^{re}\Delta, \cdot)$, where ${}^{re}\Delta$ is the set of real roots in the toroidal root system. This generalization is done by choosing a Chevalley basis for the toroidal Lie algebra which initiates Chapter 2. In Chapter 2, we restrict ourselves to 2-fold affinizations. We identify $St(\dot{\Delta}, \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}])$ with $St({}^{re}\Delta, \mathbb{C})$ and then study its structure and representations. As in the loop case, the representations usually come from *integrable* representations of the corresponding toroidal Lie algebras. We construct *universal* toroidal groups from all integrable representations of toroidal Lie algebras by using the formal trick in [KP]. Universal toroidal groups are homomorphic image of Steinberg groups. Here "universality" means that any given integrable representation of a toroidal Lie algebra can be made into a representation of the corresponding universal toroidal group. Indeed, we show in the $\dot{\Delta} = A_\ell (\ell \geq 4)$ cases that universal toroidal groups can be constructed by only using vertex representations. A main result is that in the type A_ℓ cases, we are able to give a presentation for universal toroidal groups by generators and relations. The ingredients of the proof consist of the calculations on the vertex representations, properties of Steinberg groups and a result about $K_2(\dot{\Delta}, \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}])$. It is easy to conjecture how the corresponding presentation should appear for the D_ℓ and E_ℓ cases and we point out that it will be true if the corresponding K_2 result holds. This will imply that Steinberg groups $St({}^{re}\Delta, \mathbb{C})$ and universal toroidal groups differ only in the fact that the universal toroidal groups do not measure $K_2(\mathbb{C})$.

Inside the Steinberg group $St({}^{re}\Delta, \mathbb{C})$, the subgroup $\widehat{N}$ gives rise to the toroidal

Weyl group and lower-fold toroidal Weyl groups including the finite Weyl group (0-fold) as quotient groups of $\widehat{N}$. This is ture for toroidal groups as well. We also show that the affine Kac-Moody groups are embedded in the universal toroidal groups.

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Chapter 1. Groups Graded by Finite Root Systems

§1.0. Introduction.

In this paper, the groups graded by finite root systems Δ or Δ-graded groups are introduced. These are analogues of Lie algebras graded by finite root systems which are introduced by Berman and Moody ([BM]). The background is the structures of Steinberg groups and Chevalley groups. The connection between Δ-graded groups and Steinberg groups, central extensions can be seen throughout the article.

Assume that our rings are always associative and with identity element denoted by 1, and any root system in question is one of type $A_\ell, \ell \geq 3, D_\ell, \ell \geq 4$ or $E_\ell, \ell = 6, 7, 8$, unless otherwise stated. Relative to such a root system Δ and a ring R, one defines a Steinberg group ([HO], [Mi], [Ste], [St]), denoted by $St(\Delta, R)$, by generators $\{\hat{x}_\alpha(r) : \alpha \in \Delta, r \in R\}$ and relations (cf. Definition(1.1.6)), where R is further assumed to be commutative in the cases $\Delta = D_l$, or E_l. $St(\Delta, R)$ has a collection of "one parameter" subgroups $\mathcal{X}^\alpha, \alpha \in \Delta$, which generate $St(\Delta, R)$ and satisfy $\mathcal{X}^\alpha \cap \mathcal{X}^\beta = (1)$, if $\alpha \neq \beta$. For all $\alpha \in \Delta, u \in R^\times$ (the units group of R), let $\hat{n}_\alpha(u) := \hat{x}_\alpha(u)\hat{x}_\alpha(-u^{-1})\hat{x}_\alpha(u)$ and $\hat{h}_\alpha(u) := \hat{n}_\alpha(u)\hat{n}_\alpha(-1)$. Denote by $\hat{N}$ and $\hat{H}$ the subgroups of $St(\Delta, R)$ generated by all $\hat{n}_\alpha(u)$'s and $\hat{h}_\alpha(u)$'s, respectively. Then the defining relations imply

$$(\mathcal{X}^\alpha, \mathcal{X}^\beta) \subseteq \begin{cases} (1), & \text{if } \alpha + \beta \notin \Delta \cup \{0\} \\ \mathcal{X}^{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta, \end{cases}$$

where $(\cdot, \cdot)$ denotes the group commutator. Let K be a commutative subring of R with the same identity 1 of R. For example, the subring of R generated by the identity element 1 is a commutative subring, which of course is isomorphic to one of the rings $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}, n \in \mathbb{Z}$. Let $\dot{\mathcal{X}}$ be the subgroup generated by $\hat{x}_\alpha(r), \alpha \in \Delta, r \in \mathsf{K}$ and $\dot{\mathcal{X}}^\alpha := \mathcal{X}^\alpha \cap \dot{\mathcal{X}}$. Suppose that $(\dot{G}, \dot{\phi})$ is a pair where $\dot{\phi}$ is a group homomorphism from $St(\Delta, \mathsf{K})$ onto $\dot{G}$ such that for some positive system Δ_+ of Δ, $\dot{\phi}|_{\mathcal{X}^+(\mathsf{K})}$ is one-to-one, where $\mathcal{X}^+(\mathsf{K}) = \langle \mathcal{X}(\mathsf{K})^\alpha \mid \alpha \in \Delta_+ \rangle < St(\Delta, \mathsf{K})$. Let $\mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$ be the collection of all such pairs. Then $\dot{\mathcal{X}}^\alpha \subseteq \mathcal{X}^\alpha$ and $\dot{\mathcal{X}} \in \mathfrak{S}$

where the homomorphism is natural. Recall that the conjugation by $\hat{n}_\alpha(\pm 1)$ maps $\mathcal{X}^\beta$ onto $\mathcal{X}^{r_\alpha \beta}$, where r_α is the refelction in α. If the generators and defining relations are excluded, will the properties mentioned above determine the Steinberg groups or Chevalley groups? This motivates our definition for a Δ-graded group(cf. Definition(1.2.1)). Steinberg groups and Chevalley groups are graded by finite root systems.

We show

(1.2.3)THEOREM. *Let G be a group graded by Δ. Then there is an associative ring R with 1, containing K as a subring, such that G is a homomorphic image of Steinberg group $St(\Delta, R)$. Moreover, R is commutative if Δ is of type D_ℓ or E_ℓ.*

For the proof, the critical point is to define the ring R out of such a group. The main theme of the proof is set in [BM] at the Lie algebra level.

Then for each Δ-graded group, we may attach a ring R. A Δ-homomorphism of Δ-graded groups is naturally understood to be a group homomorphism which preserves the Δ-grading. So in the category of Δ-graded groups, the morphisms involved are Δ-homomorphisms.

Considering group central extensions, we have

(1.3.2)THEOREM. *Let Δ be of type $A_\ell, \ell \geq 4, E_\ell, \ell = 6, 7, 8$ or $D_\ell, \ell \geq 5$. Any covering (U, ψ) of a Δ-graded group G is also Δ-graded and ψ is a Δ-homomorphism. Moreover there is a surjective homomorphism Ψ from $St(\Delta, R)$ onto U such that*

$$\begin{array}{ccc} St(\Delta, R) & \xrightarrow{\Psi} & U \\ & \searrow \quad \swarrow \psi & \\ & G & \end{array}$$

where R is the ring attached to G.

(1.3.3)THEOREM. *Let Δ be of type $A_l, \ell \geq 4$; $D_l, \ell \geq 5$; or $E_l, \ell = 6, 7, 8$. Let G and G' be perfect and G Δ-graded. If there is a group as a covering for both G*

and G', then G' is also Δ-graded in such a way that G and G' are Δ-homomorphic images of the same Steinberg group $St(\Delta, R)$.

The paper is organized as follows. In §1.1, some preliminary notion is presented and the set $\mathfrak{S}$ is defined. In addition, we show that for any element $(\dot{G}, \dot{\phi}) \in \mathfrak{S}$, the Weyl group of the finite root system Δ is isomorphic to the quotient group $\dot{\phi}(\hat{N})/\dot{\phi}(\hat{H})$. In §1.2, we define groups graded by finite root systems and prove Theorem(1.2.3). We show Theorems (1.3.2) and (1.3.3) in §1.3.

Conventions: in a group G, write $a^b := bab^{-1}$, the commutator $(a, b) := aba^{-1}b^{-1}$ and denote by *Int* b the conjugation by b, i.e. *Int* $b.a := a^b$. Write $H < G$ if H is a subgroup of G. $\langle \cdots \rangle$ means a (sub)group generated by $\cdots$.

The following formulas on commutators will be used later on.

$$(a, b) = (b, a)^{-1} \tag{1.0.1}$$

$$(ab, c) = (b, c)^a (a, c) \tag{1.0.2}$$

$$(a, bc) = (a, b)(a, c)^b$$

$$(ab, cd) = (b, c)^a (b, d)^{ac} (a, c)(a, d)^c \tag{1.0.3}$$

$$\big(a^c, (b, c)\big)\big(c^b, (a, b)\big)\big(b^a, (c, a)\big) = 1 \tag{1.0.4}$$

$$\big(a, (b, c)\big) = \big((a, b), c\big), \tag{1.0.5}$$

if $(a, c) = 1$, $\big((a, b), (b, c)\big) = 1$, and $\big((b, c), c\big) = 1$.

§1.1. Preliminaries.

In §1.1 and §1.2, we assume Δ is a finite indecomposable simply-laced root system of rank $\ell \geq 3$, i.e. Δ is of type A_ℓ, $\ell \geq 3$, $D_\ell, \ell \geq 4$, E_6, E_7, or E_8. Let Q be the root lattice spanned by Δ. The Weyl group invariant bilinear form on Q, normalized so that $(\alpha \mid \alpha) = 2$ for all $\alpha \in \Delta$, will be denoted by $(\cdot \mid \cdot)$. This form is positive definite. If $\alpha, \beta \in \Delta$, then $(\alpha \mid \beta)$ takes only values ± 2, ± 1 and 0,

respectively, if and only if $\alpha = \pm\beta$, $\alpha \mp \beta \in \Delta$ and $\alpha \pm \beta \notin \Delta \cup \{0\}$, respectively. For each root $\alpha \in \Delta$, the reflection in α is the linear map

$$r_\alpha : \lambda \mapsto \lambda - (\lambda \mid \alpha)\alpha,$$

on Q. Then the Weyl group, denoted by W, is generated by all the reflections r_α. In particular, W is generated by all the simple reflections r_{α_i} where $\Pi := \{\alpha_1, \cdots, \alpha_\ell\}$ is a base for the root system Δ.

Let $\mathfrak{g}$ be the simple Lie algebra over the complex field C associated with Δ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h} = \mathfrak{g}^0\}$ and $\mathfrak{g}^\alpha \neq (0)$ if and only if $\alpha \in \Delta \cup \{0\}$.

Let $\{E_\alpha, H_i : \alpha \in \Delta, i = 1, \cdots, \ell\}$ be a Chevalley basis of $\mathfrak{g}$ (see §1 of [St]). If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$ for some $c_{\alpha,\beta} \in \{\pm 1\}$. From the skew-symmetry of the Lie bracket $[\cdot, \cdot]$ and the application of the canonical anti-involution of $\mathfrak{g}$, we have formulas

$$c_{\alpha,\beta} = -c_{\beta,\alpha} \tag{1.1.1}$$

$$c_{\alpha,\beta} = -c_{-\alpha,-\beta} \tag{1.1.2}$$

We will see and use more formulas about $c_{\alpha,\beta}$'s later on. We will fix a choice of a Chevalley basis throughout this paper. In the case $\Delta = A_\ell$, a Chevalley basis is chosen as in the following example.

(1.1.3) EXAMPLE. *The description of the root system Δ of type A_ℓ is*

$$\{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq \ell + 1\}$$

where $\{\varepsilon_1, \ldots, \varepsilon_{\ell+1}\}$ is an orthonormal basis of $\mathbf{R}^{\ell+1}$. Let $\Pi = \{\alpha_i : i = 1, \cdots, \ell\}$ be a base for Δ for $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. The Weyl group is the symmetric group $S_{\ell+1}$.

The corresponding simple Lie algebra $\mathfrak{g}$ *is* $\mathfrak{sl}_{\ell+1}(\mathbf{C})$. *The set* $\{E_{ij}, i \neq j; H_i = E_{ii} - E_{i+1,i+1}, i = 1, \cdots, \ell\}$ *is a Chevalley basis of* $\mathfrak{sl}_{\ell+1}(\mathbf{C})$ *where* E_{ij} *are the standard matrix units.*

Now we give Definition(1.1.4) and Lemma(1.1.5) which are taken from [BM].

(1.1.4) DEFINITION. *An ordered pair* $(\beta, \gamma) \in \Delta \times \Delta$ *is an* A_2**-pair** *if* $(\beta \mid \gamma) = -1$. *Thus* (β, γ) *is an* A_2*-pair if and only if it is a base for an* A_2 *subroot system of* Δ. *Two* A_2*-pairs* (β, γ), (β', γ') *are* **equivalent**, *written* $(\beta, \gamma) \sim (\beta', \gamma')$, *if there is an element* w *of the Weyl group* W *of* Δ *such that* $\beta' = w\beta$, $\gamma' = w\gamma$. *The equivalence class of* (β, γ) *is denoted by* $[(\beta, \gamma)]$. *Also an ordered triple* (α, β, γ) *is called an* A_3**-triple**, *if* $\{\alpha, \beta, \gamma\}$ *forms a base of an* A_3 *root system such that* $(\alpha \mid \beta) = (\beta \mid \gamma) = -1$, *and* $(\alpha \mid \gamma) = 0$. *And we define an ordered quadruple* $(\alpha, \beta, \gamma, \delta)$ *to be* A_4**-quadruple** *in a similar way.*

(1.1.5) LEMMA.

(i) If Δ *is of type* D_ℓ *or* E_ℓ *then there is only one equivalence class of* A_2*-pairs.*

(ii) If Δ *is of type* A_ℓ *there are exactly two equivalence classes of* A_2*-pairs , which are (cf. Example(1.1.3))*

$$[(\alpha_1, \alpha_2)] = \{(\varepsilon_i - \varepsilon_j,\ \varepsilon_j - \varepsilon_k) \mid i, j, k \quad \textit{distinct}\},$$
$$[(\alpha_2, \alpha_1)] = \{(\varepsilon_i - \varepsilon_j,\ \varepsilon_k - \varepsilon_i) \mid i, j, k \quad \textit{distinct}\}.$$

We call $[(\alpha_1, \alpha_2)]$ *the class of positive* A_2*-pairs. Furthermore if* (β, γ) *is an* A_2*-pair then*

$$(\beta, \gamma) \sim (-\gamma, -\beta), \quad (\beta, \gamma) \not\sim (\gamma, \beta).$$

(iii) In all cases if (β, γ) *and* (γ, δ) *are* A_2*-pairs with* $(\beta \mid \delta) = 0$ *then*

$$(\beta, \gamma) \sim (\gamma, \delta) \sim (\beta, \gamma + \delta) \sim (\beta + \gamma, \delta).$$

□

We will call the only A_2-pair class of root systems of types D_ℓ, E_ℓ positive.

(1.1.6)DEFINITION. *Assume R is an associative ring with the identity 1. In the cases when the root system Δ is of type D_ℓ or E_ℓ, R is further assumed to be commutative. The* **Steinberg group** *is the abstract group with the following presentation:*

generators : $\hat{x}_\alpha(r); \alpha \in \Delta, r \in R.$

relations:

(R1) $\hat{x}_\alpha(r)\hat{x}_\beta(s) = \hat{x}_\alpha(r+s)$

(R2) $(\hat{x}_\alpha(r), \hat{x}_\beta(s)) = \begin{cases} 1, & \text{if } \alpha+\beta \notin \Delta \cup \{0\} \\ \hat{x}_{\alpha+\beta}(c_{\alpha,\beta} rs), & \text{if } (\alpha,\beta) \text{ is a positive } A_2\text{-pair,} \end{cases}$

where $c_{\alpha,\beta}$ is given by a fixed Chevalley basis.

(1.1.7)REMARKS.

(i) The above definition is same as that in [HO], [Mi], [Ste], or [St].

(ii) Although $c_{\alpha\beta}$ depends on the choice of a Chevalley basis, the Steinberg groups do not (up to isomorphism).

Let for $\alpha \in \Delta, u \in R^\times$(the units group of R),

$$\hat{n}_\alpha(u) := \hat{x}_\alpha(u)\hat{x}_{-\alpha}(-u^{-1})\hat{x}_\alpha(u).$$

$$\hat{h}_\alpha(u) := \hat{n}_\alpha(u)\hat{n}_\alpha(-1).$$

Then from [HO], [Mi], [Ste] and [St], we have

(R3) $\hat{n}_\alpha(u)\hat{x}_\beta(r)\hat{n}_\alpha(u)^{-1} = \hat{x}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)} r)$

(R4) $\hat{n}_\alpha(u)\hat{n}_\beta(v)\hat{n}_\alpha(u)^{-1} = \hat{n}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)} v)$

$$n_\alpha(u) = n_{-\alpha}(-u^{-1})$$

(R5) $\hat{n}_\alpha(u)\hat{h}_\beta(v)\hat{n}_\alpha(u)^{-1} = \hat{h}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)}v)\hat{h}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)})^{-1}$

(R6) $\hat{h}_\alpha(u)\hat{x}_\beta(r)\hat{h}_\alpha(u)^{-1} = \hat{x}_\beta(u^{(\beta|\alpha)}r)$

(R7) $\hat{h}_\alpha(u)\hat{n}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{n}_\beta(u^{(\beta|\alpha)}v)$

$\hat{h}_\alpha(u)^{-1} = \hat{h}_{-\alpha}(u)$

(R8) $\hat{h}_\alpha(u)\hat{h}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{h}_\beta(u^{(\beta|\alpha)}v)\hat{h}_\beta(u^{(\beta|\alpha)})^{-1}$

where $u, v \in R^\times$; $r \in R$ and elements in R appeared commute each other, where $\alpha, \beta \in \Delta$ and $\eta = \eta(\alpha, \beta)$ is such that

$$\eta(\alpha,\beta) = \begin{cases} c_{\alpha,\pm\beta}, & \text{if } (\alpha \mid \beta) = \mp 1, \\ -1, & \text{if } (\alpha \mid \beta) = \pm 2, \\ 1, & \text{if } (\alpha \mid \beta) = 0. \end{cases}$$

The following are important subgroups of $St(\Delta, R)$.

$$\hat{N} := \langle \hat{n}_\alpha(u) \mid u \in R^\times, \alpha \in \Delta \rangle$$

$$\hat{H} := \langle \hat{h}_\alpha(u) \mid u \in R^\times, \alpha \in \Delta \rangle$$

$$\mathcal{X}^\alpha := \langle \hat{x}_\alpha(r) \mid r \in R \rangle$$

$$\mathcal{X}^\pm := \mathcal{X}^\pm(\Delta_+) := \langle \mathcal{X}^\alpha \mid \alpha \in \Delta_\pm \rangle \quad \text{for a positive system } \Delta_+.$$

Let K be any commutative ring. We consider the Steinberg group $St(\Delta, \mathsf{K})$ in the rest of the section.

Take a pair (G, ϕ) where ϕ is a surjective homomorphism from $St(\Delta, \mathsf{K})$ onto a group G and $\phi|_{\mathcal{X}^+}$ is 1-1 for some positive system Δ_+. Let $\mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$ be the collection of all such pairs.

For $(G, \phi) \in \mathfrak{S}$, denote $\phi(\hat{x}_\alpha(a)), \phi(\hat{n}_\alpha(u)), \phi(\hat{h}_\alpha(u)), \phi(\hat{N}), \phi(\hat{H})$ by $x_\alpha(a)$, $n_\alpha(u)$, $h_\alpha(u)$, N, H respectively. But write $G^\alpha = \phi(\mathcal{X}^\alpha)$, $G^\pm = \phi(\mathcal{X}^\pm)$.

From (R3) and (R6), we have

$$n_\alpha(u)G^\beta n_\alpha(u)^{-1} = G^{r_\alpha\beta}. \tag{1.1.8}$$

$$h_\alpha(u)G^\beta h_\alpha(u)^{-1} = G^\beta. \tag{1.1.9}$$

(1.1.10)LEMMA. *Let $(G,\phi) \in \mathfrak{S}$. The restriction of ϕ on $\tilde{\mathcal{X}}^+ := \mathcal{X}^+(\tilde{\Delta}_+)$ relative to any positive system $\tilde{\Delta}_+$ is one to one.*

PROOF: By definition, there is a positive system Δ_+ of Δ such that ϕ is one to one on $\mathcal{X}^+$ which corresponds to Δ_+. Suppose $\tilde{\Delta}_+$ is another positive system of Δ. We need to show that ϕ is one to one on $\tilde{\mathcal{X}}^+$ which corresponds to $\tilde{\Delta}_+$. Recall that there is an element $w \in W$, such that $w(\Delta_+) = \tilde{\Delta}_+$([Ca]). Take a preimage $\hat{n} \in \hat{N}$ of w. Then for any $\hat{x} \in \mathcal{X}^+$, $\hat{n}\hat{x}\hat{n}^{-1} \in \tilde{\mathcal{X}}^+$. Now the lemma follows from the fact that $\phi(\hat{x}) = 1$ if and only if $\phi(\hat{n}\hat{x}\hat{n}^{-1}) = 1$. □

Recall that in the Steinberg group $St(\Delta, R)$, $\mathcal{X}^+ := \mathcal{X}^+$ (Δ_+) has a unique decomposition

$$\mathcal{X}^+ = \prod_{\alpha\in\Delta_+} \mathcal{X}^\alpha,$$

for an arbitrary chosen linear order on Δ_+ and each $\mathcal{X}^\alpha$ is isomorphic to the additive group $(R,+)$ (cf. [HO] and [St]). Then in the case $R = \mathsf{K}$, these facts can be passed onto G^+ for $(G,\phi) \in \mathfrak{S}$. Since the Weyl group W of Δ is a Coxeter group, then the map

$$r_\alpha \longmapsto \hat{n}_\alpha(u)\hat{H} \tag{1.1.11}$$

defines a homomorphism from W onto $\hat{N}/\hat{H}$. Morover this is an isomorphism. By means of it we will identify these two groups.

(1.1.12)LEMMA. *Let $(G,\phi) \in \mathfrak{S}$. If $\alpha \neq \beta$, then $G^\alpha \cap G^\beta = 1$.*

PROOF: Choose a positive system Δ_+ for Δ for which α is simple. If $\beta \in \Delta_+$, then we are done by Lemma(1.1.10). If $\beta \in \Delta_-\backslash\{-\alpha\}$, then α, β are in the positive system $r_\alpha(-\Delta_+)$ and we are done too. It remains to show $G^\alpha \cap G^{-\alpha} = 1$. Since $\mathcal{X}^\alpha$ is isomorphic to the additive group $(R,+)$, we have $x_\alpha(r) = 1$ if and only if $r = 0$. Suppose $x_\alpha(r) = x_{-\alpha}(s)$. Take $\gamma \in \Delta$ with $(\alpha \mid \gamma) = -1$. Then by (R2),

$$1 = \big(x_{-\alpha}(s), x_\gamma(1)\big) = \big(x_\alpha(r), x_\gamma(1)\big) = x_{\alpha+\gamma}(c_{\alpha,\gamma}r).$$

Then $r = 0$, and so $s = 0$. □

(1.1.13)LEMMA. *Let $(G, \phi) \in \mathfrak{S}$ and keep the above notation. Then $H \lhd N$ and N/H is isomorphic to the Weyl group W.*

PROOF: Since $\hat{H} \lhd \hat{N}$, then $H \lhd N$. There is a homomorphism $\psi : \hat{N}/\hat{H} \to N/H$ which factors through the composition map $\hat{N} \xrightarrow{\phi} N \to N/H$. Clearly, ψ is surjective and $\psi(\hat{n}_\alpha(r)\hat{H}) = n_\alpha(r)H$. View ψ as the map from W onto N/H. Suppose $\psi(w) = \bar{1} = H$ for $w \in W$. Write $w = r_{\beta_1} r_{\beta_2} \cdots r_{\beta_k}$ as a product of reflections. Then $\psi(\hat{n}_{\beta_1}(1)\hat{n}_{\beta_2}(1)\cdots\hat{n}_{\beta_k}(1)\hat{H}) = \bar{1}$. So, $h := n_{\beta_1}(1)n_{\beta_2}(1)\cdots n_{\beta_k}(1) \in H$. And by (1.1.9) and (1.1.8),

$$G^\alpha = hG^\alpha h^{-1} = n_{\beta_1}(1)n_{\beta_2}(1)\cdots n_{\beta_k}(1)G^\alpha\big(n_{\beta_1}(1)n_{\beta_2}(1)\cdots n_{\beta_k}(1)\big)^{-1} = G^{w\alpha}.$$

So for each $\alpha \in \Delta$, $G^{w\alpha} = G^\alpha$. Then $w\alpha = \alpha$ by the above lemma. Then $w = 1$, hence ψ is an injection and an isomorphism. □

In the Steinberg group $St(\Delta, \mathbf{K})$, let

$$\hat{N}_0 := \langle \hat{n}_\alpha(1) \mid \alpha \in \Delta \rangle \tag{1.1.14}$$

$$\hat{H}_0 := \langle \hat{h}_\alpha(-1) \mid \alpha \in \Delta \rangle. \tag{1.1.15}$$

Then

(1.1.16)LEMMA. *Let $(G, \phi) \in \mathfrak{S}$ and $N_0 := \phi(\hat{N}_0), H_0 := \phi(\hat{H}_0)$. Then*

$$\hat{N}_0/\hat{H}_0 \cong N_0/H_0 \cong W.$$

PROOF: $\hat{H}_0$ and H_0 are normal subgroups of $\hat{N}_0$ and N_0, respectively, by (R5). As in (1.1.11), $r_\alpha \mapsto \hat{n}_\alpha(u)\hat{H}_0 \mapsto n_\alpha(u)H_0$ defines a homomorphism from W onto N_0/H_0. It is an isomorphism by the same proof as that of Lemma(1.1.13). □

(1.1.17)COROLLARY. $\hat{H}_0 = \hat{N}_0 \cap \hat{H}$.

PROOF: by the second isomomorphism theorem of groups, we have

$$\frac{\hat{N}_0}{\hat{N}_0 \cap \hat{H}} \cong \frac{\hat{N}_0\hat{H}}{\hat{H}} = \frac{\hat{N}}{\hat{H}} \cong W \cong \frac{\dot{N}_0}{\dot{H}_0}.$$

Since $\hat{H}_0 \subseteq \hat{N}_0 \cap \hat{H}$, so $\hat{H}_0 = \hat{N}_0 \cap \hat{H}$. □

§1.2. Groups Graded by Finite Root Systems.

Let $\mathbf{K}$ be a commutative ring. We maintain all previous notation and terminology for $\mathfrak{S} = \mathfrak{S}(\Delta, \mathbf{K})$ and elements in $\mathfrak{S}$ (usually with overdots).

(1.2.1)DEFINITION. *A group G is called to be* **graded** *by a (finite) root system Δ (of type $A_\ell, \ell \geq 3; D_\ell, \ell \geq 4$; or E_6, E_7, E_8) or Δ-graded if there are subgroups G^α, for all $\alpha \in \Delta$ and an element $(\dot{G}, \dot{\phi}) \in \mathfrak{S}$ such that*

(Gr1) $G = \langle G^\alpha \mid \alpha \in \Delta \rangle$,

(Gr2) $\dot{G}^\alpha \subseteq G^\alpha, \alpha \in \Delta$,

(Gr3) $(G^\alpha, G^\beta) \subseteq \begin{cases} \{1\}, & \text{if } \alpha + \beta \notin \Delta \cup \{0\} \\ G^{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta, \end{cases}$

(Gr4) $G^\alpha \cap G^\beta = \{1\}$ *if* $\alpha \neq \beta$,

(Gr5) $n_\alpha(1) G^\beta n_\alpha(1)^{-1} = G^{r_\alpha \beta}$, *for* $\alpha, \beta, \in \Delta$ *with* $(\alpha \mid \beta) = -1$,

where $n_\alpha(1) = \dot{\phi}(\hat{n}_\alpha(-1)) \in \dot{G}$.

(1.2.2)EXAMPLE. *Let R be an associative ring with the identity 1. In the cases when the root system Δ is of type D_ℓ or E_ℓ, assume further that R is commutative. Then $St(\Delta, R)$ and Chevalley groups are Δ-graded.*

Our main result is

(1.2.3)THEOREM. *Let G be a group graded by Δ. Then there is an associative ring R with 1, containing $\mathbf{K}$ as a subring, such that G is a homomorphic image of*

the Steinberg group $St(\Delta, R)$. *Moreover,* R *is commutative if* Δ *is of type* D_ℓ *or* E_ℓ.

The outline of the proof is as follows. Fix a root $\alpha \in \Delta$, then G^α is abelian. Let $R = G^\alpha$, so R has an additive structure. For $r \in R$, write $x_\alpha(r)$ to be the corresponding element in G^α. Elements $x_\beta(r)$ for other roots β, can be defined since G^α and G^β are isomorphic as abelian groups. The multiplication in R comes from the commutator relations (R2) and (Gr3). Such process will make $\{x_\beta(r), \beta \in \Delta, r \in R\}$ satisfy relations (R1) and (R2). Then (Gr1) makes sure that G is a homomorphic image of $St(\Delta, R)$.

We will proceed the proof in this way.

For each root $\alpha \in \Delta$, take a set A^α having the same cardinality as G^α. Fix a bijective map log_α from G^α onto A^α. By (Gr3), G^α is abelian, thus A^α carries an additive abelian group structure by making log_α into an isomorphism. So, for $x, y \in G^\alpha$,

$$0 := log_\alpha 1$$

$$log_\alpha x + log_\alpha y := log_\alpha xy$$

$$log_\alpha(x^{-1}) = -log_\alpha x$$

Let

$$1_\alpha := log_\alpha x_\alpha(1),$$

where $x_\alpha(1) = \dot{\phi}(\hat{x}_\alpha(1)) \in \dot{G}^\alpha$.

Set $\dot{N}_0 = \dot{\phi}(\hat{N}_0)$, $\dot{H}_0 = \dot{\phi}(\hat{H}_0)$ (see (1.1.13) and (1.1.14)). By Lemma(1.1.16), the map $n_\alpha(u)\dot{H}_0 \mapsto r_\alpha$, gives an isomorphism from $\dot{N}_0/\dot{H}_0$ onto W. Denote by π the composite map from $\dot{N}_0 \to \dot{N}_0/\dot{H}_0 \to W$.

(1.2.4)LEMMA. *Let* $\alpha, \beta \in \Delta$. *Then*

(i) $G^{\alpha+\beta} = (G^\alpha, G^\beta)$, *if* $(\alpha \mid \beta) = -1$.

(ii) $nG^\beta n^{-1} = G^{w\beta}$, *where* $n \in \dot{N}_0$ *with* $\pi(n) = w \in W$.

(iii) $h_\alpha(-1)xh_\alpha(-1)^{-1} = x^{(-1)^{(\alpha|\beta)}}$, *for* $x \in G^\beta$.

PROOF: In this proof, (Gr3) is widely used. (i) Suppose $(\alpha \mid \beta) = -1$, i.e. $\alpha+\beta \in \Delta$ and $x \in G^\beta$. Since $n_\alpha(1) = n_{-\alpha}(-1) = x_{-\alpha}(-1)x_\alpha(1)x_{-\alpha}(-1)$, so by (Gr3),

$$x^{n_\alpha(1)} = x^{n_{-\alpha}(-1)} = x\big(x_{-\alpha}(-1), \big(x_\alpha(1), x\big)\big)\big(x_\alpha(1), x\big).$$

The right hand sided belongs to G_β by (Gr5). By (Gr4),

$$x\big(x_{-\alpha}(-1), \big(x_\alpha(1), x\big)\big) = 1 \tag{1.2.5}$$

$$x^{n_\alpha(1)} = \big(x_{-\alpha}(1), x\big). \tag{1.2.6}$$

(1.2.5) implies $x \in (G^{\alpha+\beta}, G^{-\alpha})$. Indeed this proves (i).

Similarly, we have

$$x^{n_\alpha(-1)} = x^{n_{-\alpha}(1)} = x\big(x_{-\alpha}(1), \big(x_\alpha(-1), x\big)\big)\big(x_\alpha(-1), x\big).$$

Applying (Gr3) to (1.2.5), we get

$$x\big(x_{-\alpha}(1), \big(x_\alpha(-1), x\big)\big) = 1 \tag{1.2.5'}$$

and hence,

$$x^{n_\alpha(-1)} = \big(x_\alpha(-1), x\big). \tag{1.2.6'}$$

So $n_\alpha(-1)G^\beta n_\alpha(1) \subset G^{\alpha+\beta}$. Putting this together with (Gr5), we get for all $\alpha, \beta, \in \Delta$ with $(\alpha \mid \beta) = \pm 1$, and $\varepsilon = \pm 1$,

$$n_\alpha(\varepsilon)G^\beta n_\alpha(-\varepsilon) = G^{r_\alpha\beta}.$$

Hence to show (ii), it suffices to show

$$n_\beta(\varepsilon)G^\beta n_\beta(-\varepsilon) = G^{-\beta}, \text{ for } \varepsilon = \pm 1.$$

By (i), $G^\beta = (G^{\alpha+\beta}, G^{-\alpha})$, for some $\alpha \in \Delta$. Then applying the conjugation by $n_\beta(\varepsilon)$ we get the result.

(iii) holds if $(\alpha \mid \beta) = 0$. Since for $\varepsilon = \pm 1$, $n_\alpha(\varepsilon) = n_{-\alpha}(-\varepsilon)$ (R4) and $h_\alpha(\varepsilon) = h_{-\alpha}(\varepsilon)^{-1}$ (R7), we may assume $(\alpha \mid \beta) = -1$ or 2. Suppose $(\alpha \mid \beta) = -1$ and $x \in G^\beta$. So by (2.1.6') and (1.2.5')

$$x^{h_\alpha(-1)} = x^{n_\alpha(-1)n_\alpha(-1)} = \big(x_\alpha(-1), x\big)^{n_\alpha(-1)}$$
$$= \big(x_{-\alpha}(1), x^{n_\alpha(-1)}\big) = \big(x_{-\alpha}(1), \big(x_\alpha(-1), x\big)\big) = x^{-1}.$$

So (iii) holds in this case. Finally we show $x^{h_\beta(-1)} = x$ for $x \in G^\beta$. Take $\alpha \in \Delta$ such that $\alpha + \beta \in \Delta$. Then by (1.2.5) and the above step, we have

$$x^{h_\beta(-1)} = \big(\big(x_\alpha(1), x\big), x_{-\alpha}(-1)\big)^{h_\beta(-1)}$$
$$= \big(\big(x_\alpha(1), x\big)^{-1}, x_{-\alpha}(-1)^{-1}\big)$$
$$= \big(\big(x_\alpha(1), x\big), x_{-\alpha}(-1)\big) = x.$$

□

For any $\alpha \in \Delta$, let W^α be the stabilizer of α in W and $\dot{N}_0^\alpha := \pi^{-1}(W^\alpha)$. Then $W^\alpha = \langle r_\beta \mid \beta \in \Delta,\ (\beta \mid \alpha) = 0\rangle$ and

$$\dot{N}_0^\alpha = \langle n_\beta(1) \mid \beta \in \Delta,\ (\beta \mid \alpha) = 0\rangle \cdot \dot{H}_0. \tag{1.2.7}$$

Take $n \in \dot{N}_0$. If $n = \prod_{i=1}^k n_{\beta_i}(\varepsilon_i)$ and $w = \pi(n), \varepsilon_i = \pm 1$, then from (R3), we have

$$n x_\alpha(1) n^{-1} = Int\, n.x_\alpha(1) = x_{w\alpha}(\varepsilon) \quad \text{for some } \varepsilon = \varepsilon(n, \alpha) \in \{\pm 1\}. \tag{1.2.8}$$

Since in $\dot{G}$, $x_\beta(r) = 1$, for $r \in \mathbf{K}$ implies $r = 0$, ε is uniquely detemined by n and α.

For any $\alpha, \beta \in \Delta$, choose an element $n \in \dot{N}_0$ such that $\pi(n)\alpha = \beta$. Assume ε is uniquely given by n and α according to (1.2.8). Define

$$\lambda_{\beta,\alpha} := \varepsilon^{-1} log_\beta . Int\, n . log_\alpha{}^{-1}. \tag{1.2.9}$$

Then by Lemma(1.2.4ii), $\lambda_{\beta,\alpha}$ is an isomorphism from A^α onto A^β such that the diagram

$$\begin{array}{ccc} G^\alpha & \xrightarrow{Int\, n} & G^\beta \\ {\scriptstyle log_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon^{-1} log_\beta} \\ A^\alpha & \xrightarrow{\lambda_{\beta,\alpha}} & A^\beta \end{array} \tag{1.2.10}$$

commutes.

(1.2.11)LEMMA. *For any $\alpha, \beta \in \Delta$, there is a unique isomorphism $\lambda_{\beta,\alpha}$ from A^α to A^β, given by (1.2.9). In other words, $\lambda_{\beta,\alpha}$ is independent of the choice of $n \in \dot{N}_0$.*

PROOF: Let $n' \in \dot{N}_0$ be another element with $\pi(n')\alpha = \beta$, and ε' be determined by n' as in (1.2.8). Note that $\pi(n^{-1}n')\alpha = \alpha$, and hence $\pi(n^{-1}n') \in W^\alpha$. So, $n^{-1}n' \in \dot{N}_0^\alpha$. Hence by (1.2.7) there are $n'' \in \langle n_\gamma(1) \mid \gamma \in \Delta, (\gamma \mid \alpha) = 0\rangle, h \in \dot{H}_0$ such that $n^{-1}n' = n''h$. By Lemma(1.2.5iii), there is $c \in \{\pm 1\}$, satisfying $Int\, h.x = x^c$ for any $x \in G^\alpha$. Note that $Int\, n'' = 1$ on G^α, since n'' commutes with G^α(Gr3). Thus for any $r \in A^\alpha$,

$$\begin{aligned}
\lambda'_{\beta,\alpha}(r) &:= \varepsilon'^{-1} log_\beta.Int\, n'.log_\alpha{}^{-1}(r) \\
&= \varepsilon'^{-1} log_\beta.Int\, n.Int\, n''.Int\, h.log_\alpha{}^{-1}(r) \\
&= \varepsilon'^{-1} log_\beta.Int\, n.(log_\alpha{}^{-1}(r))^c \\
&= \varepsilon'^{-1}\varepsilon c \lambda_{\beta,\alpha}(r)
\end{aligned}$$

Now $\lambda'_{\beta,\alpha}(1_\alpha) = \varepsilon'^{-1} log_\beta.Int\, n'.x_\alpha(1) = log_\beta.x_\beta(1) = 1_\beta$. Also $\lambda_{\beta,\alpha}(1_\alpha) = 1_\beta$. Thus $\varepsilon'^{-1}\varepsilon c = 1$, and hence $\lambda'_{\beta,\alpha} = \lambda_{\beta,\alpha}$. □

Note that the sign ε^{-1} in (1.2.9) is also uniquely determined by the fact $\lambda_{\beta,\alpha}(1_\alpha) = 1_\beta$. The following corollary is a direct consequence of the above lemma.

(1.2.12)COROLLARY.

$$\lambda_{\alpha,\beta} = \lambda_{\beta,\alpha}^{-1}$$

$$\lambda_{\gamma,\beta}\lambda_{\beta,\alpha} = \lambda_{\sim,\alpha}$$

$$\lambda_{\alpha,\alpha} = id.$$

PROOF: The maps in both sides of all three equations are of the form as in (1.2.9) with possible different signs. But the result follows from the application on 1_α. □

Now let us fix a root $\alpha \in \Delta$. Let

$$R := A^\alpha.$$

Since for each $\beta \in \Delta$, the map $t \mapsto x_\beta(t) \in \dot{G}^\beta \subseteq G^\beta$ is an injection from $(\mathbf{K}, +)$ into G^β, then the map ι_β defined by

$$t \mapsto x_\beta(t) \mapsto log_\beta x_\beta(t)$$

is an injection from $(\mathbf{K}, +)$ into A^β. We identify $\mathbf{K}$ inside A^β via ι_β. Then

$$\lambda_{\beta,\alpha} \mid_{\mathbf{K}} = id. \tag{1.2.13}$$

In fact, for any $t \in \mathbf{K}$,

$$t \mapsto \iota_\alpha(t) = log_\alpha x_\alpha(t) \overset{\lambda_{\beta,\alpha}}{\mapsto} \varepsilon^{-1} log_\beta . Int\, n\, x_\alpha(t) = \varepsilon^{-1} log_\beta x_\beta(\varepsilon t) = \iota_\beta(t) \mapsto t$$

since $n x_\alpha(t) n^{-1} = x_\beta(\varepsilon t)$, where $n \in \dot{N}_0$ is chosen so that $\pi(n)\alpha = \beta$ and ε is determined by n in (1.2.8).

For any $r \in R$, let (for the fixed root α)

$$x_\alpha(r) := log_\alpha{}^{-1} r \tag{1.2.14}$$

and for any root β,

$$x_\beta(r) := log_\beta{}^{-1}(\lambda_{\beta,\alpha}(r)). \tag{1.2.15}$$

Since $\lambda_{\alpha,\alpha} = id$, the definition of $x_\beta(r)$ is consistent with the cases when $\beta = \alpha$ or $r \in \mathbf{K}$. Then

$$\begin{aligned} x_\beta(r+s) &= log_\beta{}^{-1}(\lambda_{\beta,\alpha}(r+s)) \\ &= log_\beta{}^{-1}(\lambda_{\beta,\alpha}(r) + \lambda_{\beta,\alpha}(s)) \\ &= log_\beta{}^{-1}(\lambda_{\beta,\alpha}(r)) \cdot log_\beta{}^{-1}(\lambda_{\beta,\alpha}(s)) \\ &= x_\beta(r) x_\beta(s). \end{aligned}$$

That is

$$x_\beta(r+s) = x_\beta(r) x_\beta(s). \tag{1.2.16}$$

Consequently $x_\beta(-r) = x_\beta(r)^{-1}$.

We are ready to define a multiplication for R. For any given A_2-pair (β, γ), define a multiplication on R, $m_{(\beta,\gamma)} : R \times R \to R$ by

$$(x_\beta(r), x_\gamma(s)) = x_{\beta+\gamma}(c_{\beta,\gamma} m_{(\beta,\gamma)}(r,s)). \tag{1.2.17}$$

This definition is motivated by (R2). Note that $m_{(\beta,\gamma)}$ restricting on $\mathbf{K} \times \mathbf{K}$ is the usual multiplication in $\mathbf{K}$.

(1.2.18)LEMMA. *Let (β, γ) be an A_2-pair. Take $n \in \dot{N}_0$ such that $\pi(\beta) = \gamma$. Let $\varepsilon = \varepsilon(n, \beta)$ be determined by n as in (1.2.8). Then for $r \in R$,*

$$n x_\beta(r) n^{-1} = x_\gamma(\varepsilon r).$$

PROOF: Note both $x_\gamma(\varepsilon r)$ and $n x_\beta(r) n^{-1}$ are in G^γ (R3). Then we have

$$\begin{aligned}
n x_\beta(r) n^{-1} &= Int\, n . log_\beta{}^{-1} . \lambda_{\beta,\alpha}(r) && \text{from (1.2.15)} \\
&= log_\gamma{}^{-1} . \varepsilon\, \lambda_{\gamma,\beta} . \lambda_{\beta,\alpha}(r) && \text{from (1.2.9)} \\
&= log_\gamma{}^{-1} . \varepsilon\, \lambda_{\gamma,\alpha}(r) && \text{from (1.2.12)} \\
&= log_\gamma{}^{-1} . \lambda_{\gamma,\alpha}(\varepsilon r) \\
&= x_\gamma(\varepsilon r) && \text{from (1.2.15)}
\end{aligned}$$

□

(1.2.19)LEMMA. *Let (β, γ) be an A_2-pair. Then $m_{(\beta,\gamma)}$ is biadditive, i.e. for $r, s, t \in R$, $m := m_{(\beta,\gamma)}$,*

$$m(r + s, t) = m(r, t) + m(s, t)$$

and

$$m(r, s + t) = m(r, s) + m(r, t).$$

PROOF: From (1.2.17), (1.2.16) and (Gr3), we have

$$x_{\beta+\gamma}(c_{\beta,\gamma} m(r + s, t)) = (x_\beta(r + s), x_\gamma(t))$$

$$
\begin{aligned}
&= \big(x_\beta(r)x_\beta(s), x_\gamma(t)\big) \\
&= x_\beta(r)\big(x_\beta(s), x_\gamma(t)\big)x_\beta(r)^{-1}\big(x_\beta(r), x_\gamma(t)\big) \qquad \text{by (1.0.2)} \\
&= x_\beta(r)x_{\beta+\gamma}(c_{\beta,\gamma}m(s,t))x_\beta(r)^{-1}x_{\beta+\gamma}(c_{\beta,\gamma}m(r,t)) \\
&= x_{\beta+\gamma}(c_{\beta,\gamma}m(r,t) + c_{\beta,\gamma}m(s,t))
\end{aligned}
$$

So $m(r+s,t) = m(r,t) + m(s,t)$ and similarly $m(r,s+t) = m(r,s) + m(r,t)$. □

(1.2.20)LEMMA.

If (β,γ) and (β',γ') are two equivalent A_2-pairs of Δ, then $m_{(\beta,\gamma)} = m_{(\beta',\gamma')}$.

PROOF: Put $m = m_{(\beta,\gamma)}$ and $m' = m_{(\beta',\gamma')}$. Choose $n \in \dot{N}_0$ so that $\pi(n) = w \in W$ and $w\beta = \beta', w\gamma = \gamma'$. Suppose that

$$
\begin{aligned}
Int\, n.x_\beta(1) &= x_{\beta'}(a) \\
Int\, n.x_\gamma(1) &= x_{\gamma'}(b) \\
Int\, n.x_{\beta+\gamma}(1) &= x_{\beta'+\gamma'}(c),
\end{aligned}
$$

where $a = \varepsilon(n,\beta), b = \varepsilon(n,\gamma), c = \varepsilon(n,\beta+\gamma) \in \{\pm 1\}$ (cf. (1.2.8)). And let $\varepsilon_1 = c_{\beta,\gamma}$, $\varepsilon_2 = c_{\beta',\gamma'}$. Then by calculating the equation

$$
Int\, n.\big(x_\beta(1), x_\gamma(1)\big) = \big(Int\, n.x_\beta(1), Int\, n.x_\gamma(1)\big),
$$

we have $x_{\beta'+\gamma'}(\varepsilon_1 c) = x_{\beta'+\gamma'}(ab\varepsilon_2)$. Hence

$$
\varepsilon_2 = \varepsilon_1 c a^{-1} b^{-1}. \tag{1.2.21}
$$

Then

$$
\begin{aligned}
x_{\beta'+\gamma'}(\varepsilon_2 m'(r,s)) &= \big(x_{\beta'}(r), x_{\gamma'}(s)\big) \qquad \text{by the definition of } m' \\
&= \big(log_{\beta'}^{-1}.\lambda_{\beta',\alpha}(r), log_{\gamma'}^{-1}.\lambda_{\gamma',\alpha}(s)\big) \qquad \text{by (1.2.15)} \\
&= \big(log_{\beta'}^{-1}.\lambda_{\beta',\beta}.\lambda_{\beta,\alpha}(r),\ log_{\gamma'}^{-1}\lambda_{\gamma',\gamma}\lambda_{\gamma,\alpha}(s)\big) \qquad \text{by (1.2.12)} \\
&= \big(Int\, n.log_\beta{}^{-1}(a^{-1}\lambda_{\beta,\alpha}(r)),\ Int\, n.log_\gamma{}^{-1}(b^{-1}\lambda_{\gamma,\alpha}(r))\big) \qquad \text{by (1.2.9)}
\end{aligned}
$$

$$= \big(Int\, n.x_\beta(a^{-1}r), Int\, n.x_\gamma(b^{-1}s)\big) \qquad \text{by (1.2.15)}$$
$$= Int\, n.\big(x_\beta(a^{-1}r), x_\gamma(b^{-1}s)\big)$$
$$= Int\, n.x_{\beta+\gamma}(c_{\beta,\gamma}m(a^{-1}r, b^{-1}s)) \qquad \text{by (1.2.17)}$$
$$= x_{\beta'+\gamma'}(c\varepsilon_1 a^{-1}b^{-1}m(r,s)) \qquad \text{by (1.2.19) and (1.2.18)}$$
$$= x_{\beta'+\gamma'}(\varepsilon_2 m(r,s)) \qquad \text{by (1.2.21)}$$

Then $m'(r,s) = m(r,s)$ and $m_{(\beta,\gamma)} = m_{(\beta',\gamma')}$. □

(1.2.22)LEMMA. *Let (β,γ) be an A_2-pair. Then (γ,β) defines the opposite multiplication of $m_{(\beta,\gamma)}$. In particular, if Δ is of type D_ℓ or E_ℓ, then $m_{(\beta,\gamma)}$ is commutative.*

PROOF: Let $m = m_{(\beta,\gamma)}$ and $m' = m_{(\gamma,\beta)}$. For $r, s \in R$,

$$\begin{aligned} x_{\beta+\gamma}(c_{\beta,\gamma}m(r,s)) &= \big(x_\beta(r), x_\gamma(s)\big) \\ &= \big(x_\gamma(s), x_\beta(r)\big)^{-1} \\ &= x_{\beta+\gamma}(-c_{\gamma,\beta}m'(s,r)) \\ &= x_{\beta+\gamma}(c_{\beta,\gamma}m'(s,r)) \end{aligned}$$

where we note $-c_{\gamma,\beta} = c_{\beta,\gamma}$ from (1.1.1). Then $m(r,s) = m'(s,r)$. When Δ is of type D_ℓ or E_ℓ, then there is only one equivalence class of A_2-pairs by Lemma(1.1.5), hence $m_{(\beta,\gamma)}$ is commutative. □

(1.2.23)LEMMA. *Let (β,γ) be an A_2-pair. With respect to the multiplication $m = m_{(\beta,\gamma)}$, R is associative and with the unit element $1 = 1_\alpha$.*

PROOF: Associativity. Since rank $\Delta = \ell \geqslant 3$, there is a root δ such that (γ,δ) is an A_2-pair and $(\beta \mid \delta) = 0$. By Lemma(1.2.20) and Lemma(1.1.4), $m = m_{(\beta,\gamma)} = m_{(\beta+\gamma,\delta)} = m_{(\gamma,\delta)} = m_{(\beta,\gamma+\delta)}$. Applying the commutator relation (1.0.5) to $a = x_\beta(r), b = x_\gamma(s), c = x_\delta(t), r, s, t \in R$ and (Gr3), we have

$$\big(\big(x_\beta(r), x_\gamma(s)\big), x_\delta(t)\big) = \big(x_\beta(r), \big(x_\gamma(s), x_\delta(t)\big)\big)$$

Hence

$$(x_{\beta+\gamma}(c_{\beta,\gamma}m(r,s)), x_\delta(t)) = (x_\beta(r), x_{\gamma+\delta}(c_{\gamma,\delta}m(s,t)))$$

and

$$x_{\beta+\gamma+\delta}(c_{\beta+\gamma,\delta}m(c_{\beta,\gamma}m(r,s),t)) = x_{\beta+\gamma+\delta}(c_{\beta,\gamma+\delta}m(r,c_{\gamma,\delta}m(s,t))).$$

By calculating the identity $[[E_\beta, E_\gamma], E_\delta] = [E_\beta, [E_\gamma, E_\delta]]$ (see definition of Chevalley bases), we get

$$c_{\beta,\gamma}c_{\beta+\gamma,\delta} = c_{\beta,\gamma+\delta}c_{\gamma,\delta}.$$

Also m is biadditive by Lemma(1.2.19). Thus $m(m(r,s),t) = m(r,m(s,t))$, that is, m is associative.

Identity. We show $1 = 1_\alpha \in \mathsf{K}$ is the unit element of R. Take an element $w \in W$ such that $w\beta = \alpha$ (the fixed root). Let $\delta = w\gamma$. Then $m = m_{(\beta,\gamma)} = m_{(\alpha,\delta)}$ and

$$\begin{aligned}
&x_{\alpha+\delta}(c_{\alpha,\delta}r)\\
&= n_\alpha(1)x_\delta(r)n_\alpha(-1) \quad \text{by Lemma(1.2.18)}\\
&= x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)x_\delta(r)x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1)\\
&= x_\alpha(1)x_{-\alpha}(-1)x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))x_\delta(r)x_{-\alpha}(1)x_\alpha(-1) \quad \text{by the definition of } m\\
&= x_\alpha(1)\Big(x_{-\alpha}(-1), x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))\Big)x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))x_\delta(r)x_\alpha(-1) \quad \text{by (Gr3)}\\
&= x_\alpha(1)x_\delta\Big(-c_{-\alpha,\alpha+\delta}\big(c_{\alpha,\delta}m(1,m(1,r))\big)\Big)x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))x_\delta(r)x_\alpha(-1)
\end{aligned}$$

Now by (Gr3), $(G^\alpha, G^{\alpha+\delta}) = (G^\delta, G^{\alpha+\delta}) = 1$, thus by bring the conjugation by $x_\alpha(1)$ to the left sided hand in the last equation of above calculations, we get

$$x_\delta\Big(r - c_{-\alpha,\alpha+\delta}\big(c_{\alpha,\delta}m(1,m(1,r))\big)\Big) = x_{\alpha+\delta}\big(c_{\alpha,\delta}r - c_{\alpha,\delta}m(1,r)\big)$$

Then $r = m(1,r)$ by (Gr4). By considering $x_{\alpha+\delta}(c_{\delta,\alpha}r) = n_\delta(1)x_\alpha(r)n_\delta(-1)$ and using $m_{(\alpha,\delta)}(r,s) = m_{(\delta,\alpha)}(s,r)$, we get $r = m_{(\delta,\alpha)}(1,r) = m(r,1)$. This proves that 1 is the unit element of R with respect to the multiplication m. □

Now we can conclude Theorem(1.2.3). Here is its restatement.

(1.2.3)THEOREM. *Let G be a group graded by a finite root system Δ (of type $A_\ell, \ell \geq 3$, $D_\ell, \ell \geq 4$ or E_6, E_7, E_8) relative to an element $(\dot{G}, \dot{\phi}) \in \mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$, where K is a commutative ring. Fix any root $\alpha \in \Delta$ and let $R = G^\alpha$ as an abelian group. Relative to a Chevalley basis $\{E_\beta\}_{\beta\in\Delta} \cup \{H_i\}_{i=1}^{\ell}$ define the maps $\lambda_{\beta,\alpha} : G^\alpha \to G^\beta$ of (1.2.9) and the elements $x_\beta(r), \beta \in \Delta, r \in R$ of (1.2.15). For any A_2-pair (β, γ) define the multiplication in R by (1.2.17). Then R is an associative ring with 1, containing K as a subring, and generators $x_\beta(r)$'s satisfy relations (R1) and (R2)((1.2.16) and (1.2.17)). In particular, G is a homomorphic image of the Steinberg group $St(\Delta, R)$. In addition, R is commutative if Δ is of type D_ℓ or E_ℓ. If Δ is of type A_ℓ the two equivalence classes of A_2-pairs define opposite multiplications on R.* □

(1.2.24)REMARK. *The chosen R is independent of the choice of the root α up to isomorphism since $\lambda_{\beta,\alpha}$'s are ring isomorphisms. Thus R depends only on the choice of an equivalence class of A_2-pairs.*

§1.3. Central Extensions of Δ-graded Groups.

In this section we study central extensions of Δ-graded gorups.

Let us recall some notion about group central extensions (cf. [St]). A surjective group homomorphism ϕ from U onto G is a **central extension** of G if the kernel is contained in the center of U. A central extension (ϕ, U) of a group G is called a **covering** of G if U is *perfect*, that is, $(U, U) = U$. A central extension (ϕ, U) of a group G is called **universal** if it covers all other central extensions of G, i.e. if (ϕ', G') is any central extension of G, then there is a homomorphism $\bar{\phi}$ from U into G' such that $\phi = \phi'\bar{\phi}$. Any perfect group has a universal central extension which is unique up to isomorphism. Two perfect groups are called **centrally isogeneous** if they have same (isomorphic) universal central extension.

We will use the previous notation unless otherwise specified.

Let G be a group graded by Δ (of type $A_\ell, \ell \geq 3, D_\ell, \ell \geq 4$, or E_6, E_7, E_8) with $(\dot{G}, \dot{\phi}) \in \mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$. Then there is an associative ring R such that G is a homomorphic image of $St(\Delta, R)$. R is chosen for a fixed root with its multiplication defined by any fixed positive A_2-pair. We will simply write rs instead of $m(r,s)$ for $r, s \in R$.

(1.3.1)DEFINITION. *Let K be a commutative ring. Suppose G_1 and G_2 are Δ-graded groups relative to $(\dot{G}_1, \dot{\phi}_1), (\dot{G}_2, \dot{\phi}_2) \in \mathfrak{S}(\Delta, \mathsf{K})$, respectively. A group homomorphism σ from G_1 to G_2 is a* **Δ- homomorphism** *if $\sigma\dot{\phi}_1 = \dot{\phi}_2$.*

The following is the main result of this section.

(1.3.2)THEOREM. *Let Δ be of type $A_\ell, \ell \geq 4, E_\ell, \ell = 6, 7, 8$ or $D_\ell, \ell \geq 5$. Any covering (U, ψ) of a Δ-graded group G is also Δ-graded and ψ is a Δ-homomorphism. Moreover there is a surjective homomorphism Ψ from $St(\Delta, R)$ onto U such that*

$$\begin{array}{ccc} St(\Delta, R) & \xrightarrow{\Psi} & U \\ & \searrow \quad \swarrow \psi & \\ & G & \end{array}$$

where R is chosen as above for G.

Our proof will be constructive and the relations (R1) and (R2) play a major role in the proof. Note that G has a set of generators $\{x_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ that satisfies (R1) and (R2). We will define a set of generators $\{\bar{x}_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ in U that satisfies relations (R1) and (R2). Then U is a homomorphic image of $St(\Delta, R)$. The idea of this proof is based on that showing that $St(R) \to E(R)$(an elementary group in a Chevalley group) is a universal central extension([Mi], [St], [Ste]). Technically, [Ste] has been very helpful. Before going to the proof , we state a consequence. Again we give the proof later.

(1.3.3)THEOREM. *Let Δ be of type $A_l, \ell \geq 4; D_l, \ell \geq 5$; or $E_l, \ell = 6, 7, 8$. Let G and G' be perfect and G Δ-grded. If there is a group as a covering for both G*

and G', then G' is also Δ-graded in such a way that G and G' are Δ-homomorphic images of the same Steinberg group $St(\Delta, R)$. In particular, if G and G' are centrally isogeneous and G is graded by Δ, then G' is also graded by Δ.

The proof of Theorem(1.3.2) will be given later as a consequence of a series of preliminary results.

Let $(\tilde{U}, \psi)$ be a covering of a Δ-graded group G and C the kernel of the central extension $\psi : U \to G$.

First note that G is perfect. Indeed, for any $\alpha \in \Delta$, there is an A_2-pair (β, γ) such that $\beta + \gamma = \alpha$. Since $x_\alpha(r) = (x_\beta(1), x_\gamma(c_{\beta,\gamma} r)), G^\alpha = (G^\beta, G^\gamma)$. So $G^\alpha \subseteq (G, G)$. By (Gr1), we have $G = (G, G)$. The perfectness makes sure the existence of a covering.

The following standard lemma, sometimes called the **central trick**, is technically important. It will be used repeatedly.

(1.3.4)LEMMA. *(the central trick) Let $p : H_1 \to H_2$ be a central extension of a group H_2. If $x_1, x_2, y_1, y_2 \in H_1$ so that $px_1 = px_2$, $py_1 = py_2$, then $(x_1, y_1) = (x_2, y_2)$.* □

For any $\alpha \in \Delta$, let

$$\tilde{U}^\alpha = \psi^{-1}(G^\alpha). \tag{1.3.5}$$

Let

$$G(\alpha) := \langle G^\beta \mid \beta \in \Delta, (\beta \mid \alpha) \geqslant 0 \rangle < G. \tag{1.3.6}$$

Then $G(\alpha)$ is contained in the centralizer of G^α.

(1.3.7)LEMMA. *(i) When Δ is of type $A_\ell, \ell \geq 4$, or $E_\ell, \ell = 6, 7, 8$ for any two roots α, β with $(\alpha \mid \beta) = 0$, there are $\gamma, \delta \in \Delta$ such that $(\alpha, \gamma, \beta, \delta)$ is an A_4-quadruple.*

(ii) In $D_\ell, \ell \geqslant 5$, for any two A_2-pairs (β, γ) and (β', γ'), there exists a third

A_2-pair (β'', γ'') such that $\{\beta, \gamma, \beta', \gamma'\}$ and $\{\beta', \gamma', \beta'', \gamma''\}$ are contained in some (possibly different) $A_{\ell-1}$-subroot systems of D_ℓ.

PROOF: Examining the explicit construct of these root systems in [Bo], we may get the result. □

(1.3.8)LEMMA. *Assume Δ is of type $A_\ell, \ell \geq 4, E_\ell, \ell = 6, 7, 8$. Let $\alpha, \beta \in \Delta$ with $(\alpha \mid \beta) \geqslant 0, \alpha \neq \beta$. Then $G^\beta \subseteq (G(\alpha), G(\alpha))$.*

PROOF: We discuss two cases $(\alpha \mid \beta) = 1$ and $(\alpha \mid \beta) = 0$. In the first case, take a root γ such that $(\alpha, -\beta, \gamma)$ is an A_3-triple. Without loss of generality, we may assume $\alpha = \varepsilon_1 - \varepsilon_2, -\beta = \varepsilon_2 - \varepsilon_3, \gamma = \varepsilon_3 - \varepsilon_4$(cf. Example(1.1.1)). Then for some $c \in \{\pm 1\}$,

$$x_\beta(r) = \left(x_{\varepsilon_3 - \varepsilon_4}(cr), x_{\varepsilon_4 - \varepsilon_2}(1)\right) \in \left(G(\alpha), G(\alpha)\right),$$

since $(\varepsilon_1 - \varepsilon_2 \mid \varepsilon_3 - \varepsilon_4) = 0$ and $(\varepsilon_1 - \varepsilon_2 \mid \varepsilon_4 - \varepsilon_2) = 1$ are nonnegative. In the second case when $(\alpha \mid \beta) = 0$, we may take two roots γ, δ such that $(\alpha, \gamma, \beta, \delta)$ is an A_4-quadruple by Lemma(1.3.7). Without loss of generality again, we may assume $\alpha = \varepsilon_1 - \varepsilon_2, \gamma = \varepsilon_2 - \varepsilon_3, \beta = \varepsilon_3 - \varepsilon_4$ and $\delta = \varepsilon_4 - \varepsilon_5$. Since $(\varepsilon_1 - \varepsilon_2 \mid \varepsilon_3 - \varepsilon_5) = 0$ and $(\varepsilon_1 - \varepsilon_2 \mid \varepsilon_5 - \varepsilon_4) = 0$ are nonnegative, then for some $c \in \{\pm 1\}$,

$$x_\beta(r) = \left(x_{\varepsilon_3 - \varepsilon_5}(cr), x_{\varepsilon_5 - \varepsilon_4}(1)\right) \in \left(G(\alpha), G(\alpha)\right).$$

So the result follows. □

(1.3.9)LEMMA. *Assume Δ is of type $A_\ell, \ell \geq 4, E_\ell, \ell = 6, 7, 8$. Let $\alpha, \beta \in \Delta$ and $\alpha \neq \beta$. If $(G^\alpha, G^\beta) = 1$, then $(\tilde{U}^\alpha, \tilde{U}^\beta) = 1$(cf. (1.3.5)).*

PROOF: $(G^\alpha, G^\beta) = 1$ implies $(\alpha \mid \beta) \geqslant 0$. Then $G^\beta \subseteq G(\alpha)$. Let $\tilde{x}, \tilde{y}$ be arbitrary preimages of $x \in G^\alpha$ and $y \in G(\alpha)$ respectively. Then by Lemma(1.3.4), $(\tilde{x}, \tilde{y})$ depends only on x, y. Furthermore, $(\tilde{x}, \tilde{y})$ is in C since $(\psi(\tilde{x}, \tilde{y})) = (x, y) = 1$. Define a map λ_x from $G(\alpha)$ to C by $\lambda_x(y) := (\tilde{x}, \tilde{y})$, where $\tilde{y}$ is any preimage of y. Since C is central, we see from (1.0.2) that λ_x is a group homomorphism and

hence hence $\lambda_x\{(G(\alpha),G(\alpha))\} = 1$. But $G^\beta \subseteq (G(\alpha),G(\alpha))$, so $\lambda_x(G^\beta) = 1$. Thus $(\tilde{x},\tilde{U}) = 1$. But $\tilde{x}$ is arbitrary, and hence $(\tilde{U}^\alpha,\tilde{U}^\beta) = 1$. □

For the generators $x_\alpha(r), r \in R, \alpha \in \Delta$, let $y_\alpha(r) \in U$ be *any* preimage of $x_\alpha(r)$. For $\alpha \in \Delta$, choose any two roots β,γ with $\alpha = \beta + \gamma$. Define

$$\bar{x}_\alpha(r) := (y_\beta(c_{\beta,\gamma}r), y_\gamma(1)). \tag{1.3.10}$$

So by the central trick

$$\bar{x}_\alpha(r) = (\bar{x}_\beta(c_{\beta,\gamma}r), \bar{x}_\gamma(1)). \tag{1.3.11}$$

Let

$$U^\alpha = \{\bar{x}_\alpha(r) \mid r \in R\}.$$

(1.3.12)LEMMA. *Let Δ be of type $A_\ell, \ell \geq 4, E_\ell, \ell = 6,7,8$ or $D_\ell, \ell \geq 5$. Then $\bar{x}_\alpha(r)$ is independent of the choice and the order of β,γ.*

PROOF FOR A_ℓ OR E_ℓ: Independence of the choice. Suppose $\alpha = \beta' + \gamma'$ is another such representation of α with $\{\beta,\gamma\} \neq \{\beta',\gamma'\}$ (set-theoretic).

$$1 = (\alpha \mid \beta) = (\beta' + \gamma' \mid \beta) = (\beta' \mid \beta) + (\gamma' \mid \beta).$$

So, either

$$(\beta' \mid \beta) = 1, (\gamma' \mid \beta) = 0$$

or

$$(\beta' \mid \beta) = 0, (\gamma' \mid \beta) = 1.$$

We study these cases separately.

Case 1: $(\beta' \mid \beta) = 1, (\gamma' \mid \beta) = 0$. We may and will apply the commutator formula (1.0.5) with $a = y_\beta(\varepsilon r), b = y_{\beta'-\beta}(\eta\varepsilon\varepsilon'), c = y_{\gamma'}(1)$ where $\varepsilon = c_{\beta,\gamma}, \varepsilon' = c_{\beta',\gamma'}, \eta = c_{\beta,\beta'-\beta}$, because we have $(\beta \mid \gamma') = 0, (\beta' \mid \beta' - \beta + \gamma) = 0, (\gamma' \mid \beta' - \beta + \gamma) = 0$ and Lemma(1.3.9). So

$$(y_{\beta'}(\varepsilon' r), y_{\gamma'}(1)) = ((y_\beta(\varepsilon r), y_{\beta'-\beta}(\eta\varepsilon\varepsilon')), y_{\gamma'}(1))$$

$$= (y_\beta(\varepsilon r), (y_{\beta'-\beta}(\eta\varepsilon\varepsilon'), y_{\gamma'}(1)))$$
$$= (y_\beta(\varepsilon r), y_\gamma(\eta\eta'\varepsilon\varepsilon'))$$

where $\eta' = c_{\gamma-\gamma',\gamma'}$, and the central trick has been applied. Now the following calculation yields that $\eta\eta'\varepsilon\varepsilon' = 1$:

$$c_{\beta',\gamma'}c_{\beta,\beta'-\beta}E_{\beta'+\gamma'} = [[E_\beta, E_{\beta'-\beta}], E_{\gamma'}]$$
$$= [E_\beta, [E_{\beta'-\beta}, E_{\gamma'}]]$$
$$= c_{\beta,\gamma}c_{\gamma-\gamma',\gamma'}E_{\beta+\gamma}.$$

So $\bar{x}_\alpha(r)$ is independent of the choice of β, γ in this case.

Case 2: $(\beta' \mid \beta) = 0, (\gamma' \mid \beta) = 1$. By using Lemma(1.3.9), and then (1.0.5) and the central trick, we have for any $r \in R$,

$$(y_{\beta'}(\varepsilon' r), y_{\gamma'}(1)) = ((y_\gamma(-1), y_{\beta'-\gamma}(-\eta\varepsilon' r)), y_{\gamma'}(1))$$
$$= (y_\gamma(-1), (y_{\beta'-\gamma}(-\eta\varepsilon' r), y_{\gamma'}(1)))$$
$$= (y_\gamma(-1), y_\beta(-\eta\eta'\varepsilon' r))$$
$$= (y_\gamma(1)^{-1}, y_\beta(\eta\eta'\varepsilon' r)^{-1}) \quad \text{by the central trick}$$
$$= (y_\beta(\eta\eta'\varepsilon' r), y_\gamma(1))$$

Where $\varepsilon = c_{\beta,\gamma}, \varepsilon' = c_{\beta',\gamma'}, \eta = c_{\gamma,\beta'-\gamma}, \eta' = c_{\beta,-\gamma',\gamma'}$. Also Jacobi identity,

$$[[E\gamma, [E_{\beta'-\gamma}], E_{\gamma'}] = [E\gamma, [E_{\beta'-\gamma}, E_{\gamma'}]]$$

and (1.1.1) imply $\varepsilon = -\varepsilon'\eta\eta'$. Then it follows that $\bar{x}_\alpha(r)$ is independent of the choice of β, γ.

We still have to show that $\bar{x}_\alpha(r)$ is independent of the order of β, γ. By examining Example(1.1.1), we have the fact that in an $A_\ell, \ell \geqslant 3$, there are at least two distinct representations $\alpha = \beta + \gamma = \beta' + \gamma'$. We choose such a pair $\{\beta', \gamma'\}$ not equal to $\{\beta, \gamma\}$ as sets. Then by the independence of choice,

$$\bar{x}_\alpha(r) = (y_\beta(c_{\beta,\gamma}r), y_\gamma(1))$$

$$= \big(y_{\beta'}(c_{\beta',\gamma'}r), y_{\gamma'}(1)\big)$$
$$= \big(y_{\gamma}(c_{\gamma,\beta}r), y_{\beta}(1)\big).$$

This shows that $\bar{x}_\alpha(r)$ is independent of the order of β, γ.

PROOF FOR D_ℓ: For $\ell \geqslant 5$, $\Delta = B_\ell$ contains two subroot systems of type $A_{\ell-1}(\ell - 1 \geqslant 4)$, whose union contains a base for Δ and whose intersection is an $A_{\ell-2}$ subroot system of Δ.

With this observation, we see from Lemma(1.3.7) that given two representations $\alpha = \beta' + \gamma' = \beta + \gamma$, we may always find a third distinct representation of $\alpha = \beta'' + \gamma''$ such that $\{\beta, \gamma, \beta'', \gamma''\}$ and $\{\beta', \gamma', \beta'', \gamma''\}$ each lie in an $A_{\ell-1}(\ell - 1 \geqslant 4)$ subroot system of Δ. Then we can apply the result for $A_{\ell-1}$ $(\ell - 1 \leq 4)$, and get

$$\bar{x}_\alpha(r) = \big(y_\beta(c_{\beta,\gamma}r), y_\gamma(1)\big)$$
$$= \big(y_{\beta''}(c_{\beta'',\gamma''}r), y_{\gamma''}(1)\big)$$
$$= \big(y_{\beta'}(c_{\beta',\gamma'}r), y_{\gamma'}(1)\big).$$

□

(1.3.13)LEMMA. *Let Δ be of type $A_\ell, \ell \geq 4, E_\ell, \ell = 6, 7, 8$ or $D_\ell, \ell \geq 5$. The generators $\bar{x}_\alpha(r), r \in R, \alpha \in \Delta$ satisfy the relations (R1) and (R2).*

PROOF FOR A_ℓ OR E_ℓ: Let $\alpha = \beta + \gamma, \alpha, \beta, \gamma \in \Delta$. We use the notation defined in (1.3.10). Then for $\varepsilon = c_{\beta,\gamma}$,

$$\bar{x}_\alpha(r+s) = \big(y_\beta(c_{\beta,\gamma}(r+s)), y_\gamma(1)\big)$$
$$= \big(y_\beta(c_{\beta,\gamma}r)y_\beta(c_{\beta,\gamma}s), y_\gamma(1)\big) \quad \text{by the central trick}$$
$$= y_\beta(c_{\beta,\gamma}r)\big(y_\beta(c_{\beta,\gamma}s), y_\gamma(1)\big)y_\beta(c_{\beta,\gamma}r)^{-1}\big(y_\beta(c_{\beta,\gamma}r), y_\gamma(1)\big) \quad \text{by (1.0.2)}$$
$$= \big(y_\beta(c_{\beta,\gamma}s), y_\gamma(1)\big)\big(y_\beta(c_{\beta,\gamma}r), y_\gamma(1)\big)$$

$$= \bar{x}_\alpha(s)\bar{x}_\alpha(r)$$
$$= \bar{x}_\alpha(r)\bar{x}_\alpha(s)$$

So (R1) holds for $\bar{x}_\alpha(r)$'s.

If $(\alpha \mid \delta) \geqslant 0$, then by Lemmas(1.3.8) and (1.3.9) and the central trick we have

$$\big(\bar{x}_\alpha(r), \bar{x}_\delta(s)\big) = 1.$$

Now we assume $(\alpha \mid \delta) = -1$ and (α, δ) is a positive A_2-pair. It remains to show

$$\big(\bar{x}_\alpha(r), \bar{x}_\delta(s)\big) = \bar{x}_{\alpha+\delta}(c_{\alpha,\delta} rs).$$

Case 1: $\Delta = A_\ell$. We need to show

$$\big(\bar{x}_{ij}(r), \bar{x}_{jk}(s)\big) = \bar{x}_{ik}(rs) \quad \text{i, j, k distinct.}$$

where $\bar{x}_{ij}(r) := \bar{x}_{\varepsilon_i - \varepsilon_j}(r)$, etc. (cf. Lemma(1.1.5)).

Take m not equal to i, j, k. Then applications of the central trick and (1.0.5) (see (1.3.11) as well) yield

$$\begin{aligned}
\big(\bar{x}_{ij}(r), \bar{x}_{jk}(s)\big) &= \big(\bar{x}_{ij}(r), \big(\bar{x}_{jm}(s), \bar{x}_{mk}(1)\big)\big) \\
&= \big(\big(\bar{x}_{ij}(r), \bar{x}_{jm}(s)\big), \bar{x}_{mk}(1)\big) \\
&= \big(\bar{x}_{im}(rs), \bar{x}_{mk}(1)\big) \\
&= \big(y_{im}(rs), y_{mk}(1)\big) \\
&= \bar{x}_{ik}(rs).
\end{aligned}$$

Case 2: $\Delta = E_\ell$. There is only one class of positive A_2-pairs. Choose $\beta' \in \Delta$ such that (α, δ, β') is A_3-triple. Then $\beta := \delta + \beta'$ satisfies $(\beta \mid \delta) = 1$ and $(\beta \mid \alpha) = -1$. Then $\gamma := \delta - \beta, \alpha + \beta \in \Delta$. Then

$$\big(\bar{x}_\alpha(r), \bar{x}_\delta(s)\big) = \big(\bar{x}_\alpha(r), (\bar{x}_\beta(\varepsilon s), \bar{x}_\gamma(1))\big)$$

$$= ((\bar{x}_\alpha(r), \bar{x}_\beta(\varepsilon s)), \bar{x}_\gamma(1))$$
$$= (\bar{x}_{\alpha+\beta}(\varepsilon' \varepsilon r s), \bar{x}_\gamma(1))$$
$$= \bar{x}_{\alpha+\beta+\gamma}(\eta \varepsilon \varepsilon' r s)$$

where $\varepsilon = c_{\beta,\gamma}, \varepsilon' = c_{\alpha,\beta}, \eta = c_{\alpha+\beta,\gamma}$. As before, we have $\eta\varepsilon\varepsilon' = c_{\alpha,\delta}$. So (R2) is satisfied by $\bar{x}_\alpha(r)$'s.

PROOF FOR D_ℓ:

We will need an A_4-quatruple (cf. Lemma(1.3.7)). That is the reason why we assume $\ell \geq 5$.

The relation (R1) follows from the same proof as that for A_ℓ and E_ℓ. For (R2), suppose $\alpha, \delta \in \Delta$, $r, s \in R$.

If $(\alpha \mid \delta) = 2$, i.e. $\alpha = \delta$, then from (R1), $(\bar{x}_\alpha(r), \bar{x}_\alpha(s)) = 1$.

If $(\alpha \mid \delta) = 1$, Lemmas(1.3.8) and (1.3.9) hold for α, δ by replacing α, β there. Using $(G^\alpha, G^\delta) = 1$, we have $(\bar{x}_\alpha(r), \bar{x}_\delta(s)) = 1$.

If $(\alpha \mid \delta) = -1$, we know that $\{\alpha, \delta\}$ can be imbedded into a subroot system of type $A_{\ell-1}$, then $(\bar{x}_\alpha(r), \bar{x}_\alpha(s)) = \bar{x}_{\alpha+\delta}(c_{\alpha,\delta} r s)$.

Finally, assume $(\alpha \mid \delta) = 0$. We will use the explicit construction for the root system $\Delta = D_l$. That is

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq \ell\}$$

where $\{\varepsilon_i\}$ is the standard basis of $\mathbf{R}^\ell$ ([Bo]).

Recall that Δ has only one W-orbit of roots. Without loss of generality, assume $\alpha = \varepsilon_1 - \varepsilon_2$. Then δ is one of the following roots.

$$\{\pm(\varepsilon_1 + \varepsilon_2), \pm\varepsilon_i \pm \varepsilon_j, 3 \leq i \neq j \leq \ell\}.$$

We claim that if $\delta \in \{\pm\varepsilon_i \pm \varepsilon_j, 3 \le i \ne j \le \ell\}$, there exist two roots $\beta, \gamma \in \Delta$ such that $(\alpha, \beta, \delta, \gamma)$ is an A_4-quadruple. Indeed, if $\delta = \varepsilon_i + \varepsilon_j$, take $\beta = \varepsilon_2 - \varepsilon_i, \gamma = \varepsilon_k - \varepsilon_j$; if $\delta = \varepsilon_i - \varepsilon_j$, take $\beta = \varepsilon_2 - \varepsilon_i, \gamma = \varepsilon_j - \varepsilon_k$; if $\delta = -\varepsilon_i + \varepsilon_j$, take $\beta = \varepsilon_2 + \varepsilon_i, \gamma = -\varepsilon_j + \varepsilon_k$; if $\delta = -\varepsilon_i - \varepsilon_j$, take $\beta = \varepsilon_2 + \varepsilon_i, \gamma = \varepsilon_j + \varepsilon_k$ where $3 \le i, j, k \le \ell$ and i, j, k are distinct. Also it is clear from the point of view of Weyl group since $W = S_\ell \ltimes 2^{\ell-1}$ where S_ℓ is the symmetric group on ℓ letters and $2^{\ell-1}$ consists of even number of sign changes. Then applying the result on $A_\ell (\ell \geqslant 4)$, we get $(\bar{x}_\alpha(r), \bar{x}_\delta(s)) = 1$ for $\delta \in \{\pm\varepsilon_i \pm \varepsilon_j, 3 \le i \ne j \le \ell\}$ $(\alpha = \varepsilon_1 - \varepsilon_2)$.

It remains to show $(\bar{x}_\alpha(r), \bar{x}_\delta(s)) = 1$ for $\delta = \pm(\varepsilon_1 + \varepsilon_2)$ $(\alpha = \varepsilon_1 - \varepsilon_2)$. Applying the conjugation by $\bar{n}_\delta(1)$ and the central trick, we need only to prove $(\bar{x}_\alpha(r), \bar{x}_\delta(s)) = 1$ for $\delta = \varepsilon_1 + \varepsilon_2$.

Let $\beta = -\varepsilon_1 + \varepsilon_3$. Then (α, β, δ) is an A_3-triple. Note that $(\bar{x}_\alpha(r), \bar{x}_\delta(s))$ is central since $\psi(U^\alpha, U^\delta) = (G^\alpha, G^\delta) = 1$. Then

$$
\begin{aligned}
&(\bar{x}_\alpha(r), \bar{x}_\delta(s)) \\
&= (\bar{x}_\alpha(r), \bar{x}_\delta(s))^{\bar{x}_\beta(1)} \\
&= \big((\bar{x}_\beta(1), \bar{x}_\alpha(r))\bar{x}_\alpha(r), (\bar{x}_\beta(1), \bar{x}_\delta(r))\bar{x}_\delta(s)\big) \\
&= \big(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r)\bar{x}_\alpha(r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s)\bar{x}_\delta(s)\big) \\
&= (\bar{x}_\alpha(r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))^{\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r)} \cdot (\bar{x}_\alpha(r), \bar{x}_\delta(s))^{\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r)\bar{x}_{\beta+\delta}(c_{\beta,\delta}s)} \\
&\quad \cdot (\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s)) \cdot (\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_\delta(s))^{\bar{x}_{\beta+\delta}(c_{\beta,\delta}s)} \quad \text{by (1.0.3)} \\
&= \bar{x}_{\alpha+\beta+\delta}(c_{\alpha,\beta+\delta}c_{\beta,\delta}rs) \cdot (\bar{x}_\alpha(r), \bar{x}_\delta(s)) \\
&\quad \cdot (\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s)) \cdot \bar{x}_{\alpha+\beta+\delta}(c_{\alpha,\beta+\delta}c_{\beta,\delta}rs).
\end{aligned}
$$

Note the middle two terms of last expression are central. Again by calculation Jacobi identity, $[[E_\beta, E_\alpha], E_\delta] = -[E_\alpha, [E_\beta, E_\delta]]$, we have $c_{\alpha,\beta+\delta}c_{\beta,\delta} = -c_{\alpha,\beta+\delta}c_{\beta,\delta}$. Then $(\bar{x}_\alpha(r), \bar{x}_\delta(s)) = (\bar{x}_\alpha(r), \bar{x}_\delta(s))(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))$, and

$$(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s)) = 1.$$

Since r, s are arbitrary, then

$$(\bar{x}_{\beta+\alpha}(r), \bar{x}_{\beta+\delta}(s)) = 1 \quad \text{for all } r, s \in R.$$

Now applying the conjugation by $\bar{n}_\beta(1)$ and the central trick, we have

$$(\bar{x}_\alpha(r), \bar{x}_\delta(s)) = 1 \quad \text{for all } r, s \in R.$$

This completes our proof. □

(1.3.14)LEMMA. *$U = \langle U^\alpha \mid \alpha \in \Delta \rangle$. Hence (Gr1) holds for U.*

PROOF: Let $U' := \langle U^\alpha \mid \alpha \in \Delta \rangle$ and C be the kernel of ψ from U onto G. Since $\psi(U') = G$, then $U = U'C$. Then

$$U = (U, U) = (U'C, U'C) = (U', U') = U'$$

where the last equality follows from the relations (R1) and (R2) for $\bar{x}_\alpha(r)$'s. □

Now let us show Theorem(1.3.2).

PROOF OF THEOREM(1.3.2): Up to now, we have established a surjective homomorphism Ψ from $St(\Delta, R)$ onto U by sending $\hat{x}_\alpha(r)$ to $\bar{x}_\alpha(r)$. Let $\dot{U}$ be the subgroup generated by $\{\bar{x}_\alpha(r) \mid \alpha \in \Delta, r \in \mathsf{K}\}$ and $\dot{U}^\alpha = \{\bar{x}_\alpha(r) \mid r \in \mathsf{K}\}$ for each $\alpha \in \Delta$. We show $\dot{U} \in \mathfrak{S}$.

Clearly, $\hat{x}_\alpha(r) \to \bar{x}_\alpha(r)$ defines a surjective homomorphism, denoted by $\dot{\phi}_u$, from $St(\Delta, \mathsf{K})$ onto $\dot{U}$. So the diagram commutes.

$$\begin{array}{ccc} St(\Delta, \mathsf{K}) & \xrightarrow{\dot{\phi}_u} & \dot{U} \\ \| & & \downarrow \Psi \\ St(\Delta, \mathsf{K}) & \xrightarrow{\dot{\phi}} & \dot{G} \end{array}$$

Since $\dot{G} \in \mathfrak{S}$, so $\dot{\phi}$ restricting on $\mathcal{X}^+$ relative to a positive system Δ_+ of Δ is an isomorphism. The commutative diagram implies $\dot{\phi}_u$ restricting on $\mathcal{X}^+$ is an isomorphism as well. This implies $(\dot{U}, \dot{\phi}_u) \in \mathfrak{S}$.

It remains to verify the axioms (Gr1) to (Gr5). (Gr1) is from Lemma(1.3.14). (Gr2) is clear from the definition of $\dot{U}^\alpha$. (Gr3) follows from Lemma(1.3.13). (Gr4) holds for U since it holds for $St(\Delta, R)$ and G. (Gr5) is from the relation (R3). It is clear from the construction that ψ is a Δ-homomorphism, and the theorem is proved. □

PROOF OF THEOREM(1.3.3): Let R be the associative ring relative to G. Let $\psi(\psi') : U \to G(G')$ be the universal central extension of $G(G')$. By Theorem(1.3.2), U is graded by Δ. Moreover, the set generators $\{x_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ in G can be lifted to a set of generators $\{\bar{x}_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ in U which satisfies relations (R1) and (R2). Denote the element in $\mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$ relative to G (U, respectively) by $(\dot{G}, \dot{\phi})$ ($(\dot{U}, \dot{\phi}_u)$, respectively). The meanings of $U^\alpha, U^\pm$ (relative to a positive system of Δ), $\bar{n}_\alpha(u), \bar{h}_\alpha(u), \dot{U}_\alpha, \dot{U}^\pm$, etc. are defined as before in a obvious manner. Pass these objects to G' by the central extension homomorphism ψ', for example, $G'^\alpha := \psi'(U^\alpha), x'_\alpha(r) := \psi'(\bar{x}_\alpha(r)), \dot{G}' := \psi'(\dot{U}), \dot{G}'^\pm = \psi'(\dot{U}^\pm)$, etc. Then $\dot{\phi}' := \psi'\dot{\phi}_u$ is homomorphism from $St(\Delta, \mathsf{K})$ onto $\dot{G}'$. We will show that G' is Δ-graded relative to $(\dot{G}', \dot{\phi}')$. So it suffices to show that $(\dot{G}', \dot{\phi}') \in \mathfrak{S} = \mathfrak{S}(\Delta, \mathsf{K})$ and that the axiom (Gr4) holds, since the other axioms are direct consequences of relations (R1) and (R2) and the fact that ψ' is a homomorphism.

To be clear, we describe the relations of above maps by the following commutative diagrams with the generators:

$$\begin{array}{ccc} St(\Delta, R);\ \hat{x}_\alpha(r) & \longrightarrow & G;\ x_\alpha(r) \\ \downarrow & \searrow & \uparrow{\scriptstyle \psi} \\ G';\ x'_\alpha(r) & \underset{\psi'}{\longleftarrow} & U;\ \bar{x}_\alpha(r) \end{array}$$

and

$$\begin{array}{ccc} St(\Delta, \mathsf{K});\ \hat{x}_\alpha(t) & \xrightarrow{\dot{\phi}} & \dot{G};\ x_\alpha(t) \\ {\scriptstyle \dot{\phi}'}\downarrow & \searrow^{\dot{\phi}_u} & \uparrow{\scriptstyle \psi} \\ \dot{G}';\ x'_\alpha(t) & \underset{\psi'}{\longleftarrow} & \dot{U};\ \bar{x}_\alpha(t). \end{array}$$

Now arbitrarily fix a positive system Δ_+ of Δ. Then $\dot{\phi}_u|_{\mathcal{X}^+(\mathbf{K})}$ is injective since $(\dot{U}, \dot{\phi}_u) \in \mathfrak{S}$ by Theorem(1.3.2). Recall that the center of $\mathcal{X}^+(\mathbf{K})$ is trivial([HO], [St]). Then the center of $\dot{U}_+$ is trivial. Suppose $\hat{x} \in \mathcal{X}^+(\mathbf{K}) \cap ker(\dot{\phi}')$. Then $\dot{\phi}_u(\hat{x}) \in \dot{U}^+ \cap ker\psi'$. $ker\psi'$ is central by hypothesis. So $\dot{\phi}_u(\hat{x}) = 1$, and $\hat{x} = 1$. So $\dot{\phi}'|_{\mathcal{X}^+(\mathbf{K})}$ is injective. This proves $(\dot{G}', \dot{\phi}') \in \mathfrak{S}$.

We show that $G'^{\alpha} \cap G'^{\beta} = 1$, if $\alpha \neq \beta$. Let $x'_\alpha(r) = x'_\beta(s)$. Then

$$\bar{x}_\alpha(r) = \bar{x}_\beta(s)z$$

for some $z \in ker\psi' \subseteq Cent(U)$. It suffices to show $r = s = 0$. We need to consider 4 cases: $\alpha = -\beta$; $(\alpha \mid \beta) = -1$; $(\alpha \mid \beta) = -1$; and $(\alpha \mid \beta) = 0$.

If $\alpha = -\beta$, take $\gamma \in \Delta$ so that $(\alpha \mid \gamma) = -1$. Then by (R2),

$$1 = \big(\bar{x}_{-\alpha}(s)z, \bar{x}_\gamma(1)\big) = \big(\bar{x}_\alpha(r), x_\gamma(1)\big) = x_{\alpha+\gamma}(c_{\alpha,\gamma}r).$$

Then $r = 0$ and $s = 0$.

If $(\alpha \mid \beta) = -1$, i.e. $\alpha + \beta \in \Delta$, then

$$1 = \big(\bar{x}_\alpha(r), \bar{x}_\alpha(1)\big) = \big(\bar{x}_\beta(s)z, \bar{x}_\alpha(1)\big) = \bar{x}_{\beta+\alpha}(c_{\beta,\alpha}s).$$

Then $s = r = 0$.

If $(\alpha \mid \beta) = 1$, we take $\gamma \in \Delta$ so that $(\alpha, -\beta, \gamma)$ is an A_3-triple. Then

$$1 = \big(\bar{x}_\alpha(r), \bar{x}_{-\gamma}(1)\big) = \big(\bar{x}_\beta(s)z, \bar{x}_{-\gamma}(1)\big) = \bar{x}_{\beta-\gamma}(c_{\beta,-\gamma}s).$$

So $s = 0$ and $r = 0$.

Finally if $(\alpha \mid \beta) = 0$, then there exists a third root γ so that (α, γ, β) is an A_3-triple. Then

$$\bar{x}_{\alpha+\gamma}(c_{\alpha,\gamma}r) = \big(\bar{x}_\alpha(r), \bar{x}_\gamma(1)\big) = \big(\bar{x}_\beta(s)z, \bar{x}_\gamma(1)\big) = \bar{x}_{\beta+\gamma}(c_{\beta,\gamma}s).$$

But $(\alpha + \gamma \mid \beta + \gamma) = 1$. Then $r = s = 0$ follows from the third case. So (Gr4) holds and the result follows. □

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Chapter 2. Toroidal Groups

§2.0. Introduction.

The theory of affine Lie algebras, as one type of Kac-Moody algebras, and their representations is well developed and has many applications in other fields, for examples, modular forms, character theory, conformal field theory, quantum mechanics and so on. A simply laced affine Lie algebra $\hat{\mathfrak{g}}$ has a realization of being a universal central extension of the loop algebra $\mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}}\dot{\mathfrak{g}}$, where $\dot{\mathfrak{g}}$ is the finite dimensional simple complex Lie algebra. Loosely speaking, $\mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}}\dot{\mathfrak{g}}$ is a 1-fold affinization (only one variable in the Laurent polynomial ring) of $\dot{\mathfrak{g}}$. Now the 2-fold affinization of $\dot{\mathfrak{g}}$ is the Lie algebra $R_2\otimes\dot{\mathfrak{g}}$ for $R_2=\mathbb{C}[s^{\pm1},t^{\pm1}]$. Moody, Rao and Yokonuma [MRY] constructed a great number of representations for the universal central extension $\mathfrak{t}$ (called a toroidal Lie algebra) of $R_2\otimes\dot{\mathfrak{g}}$ (for simply laced $\dot{\mathfrak{g}}$) through vertex operators. These representations, called vertex representations, are *integrable* to an action of a group which will be called a toroidal group (arising from vertex representations).

The general methodology of constructing groups from Lie algebras was used in [Ch1], [Ch2] and [De]. Using a Chevalley basis of $\dot{\mathfrak{g}}$, Chevalley and Demazure constructed an affine group scheme over $\mathbb{Z}$. More precisely, they proved that for any complex semisimple Lie group $G_{\mathbb{C}}$, there is exactly one Chevalley group scheme $\mathcal{G}$, i.e. a certain group functor on the category of all rings, such that $\mathcal{G}(\mathbb{C})$ is isomorphic to $G_{\mathbb{C}}$. The problem of attaching groups to Kac-Moody algebras was first studied by Moody and Teo [MT]. For historical notes and bibliography of later work on such groups, see [Ti2]. Basically, the methods used to define such groups are free of external choices. Kac and Peterson [KP] achieved this independence by a formal trick by considering all *integrable* representations of the Kac-Moody Lie algebras. Also Tits [Ti1] formulated the axiomatic definition of Kac-Moody groups, which is similar to that of Chevalley and Demazure.

In this paper, we

1. Construct (universal) toroidal groups (for simply laced $\dot{\mathfrak{g}}$) over $\mathbb{C}$ by using all *integrable* representations of $\mathfrak{t}$.

2. Show that they have relations similar to the *defining* relations of Kac-Moody groups.

3. Prove that these relations are the defining relations of toroidal groups in the $\dot{\mathfrak{g}} = A_\ell, \ell \geq 4$ cases, and make a conjecture for other cases.

4. Study corresponding Steinberg groups and Weyl groups.

To formulate our problem, we view $R_2 \otimes \dot{\mathfrak{g}}$ as a Lie algebra affinized twice from $\dot{\mathfrak{g}}$ by the lowest root of $\dot{\mathfrak{g}}$. As the universal central extension of $R_2 \otimes \dot{\mathfrak{g}}$, $\mathfrak{t}$ is graded by a *toroidal root system* $\Delta = \dot{\Delta} \cup \{0\} + \mathbf{Z}\delta_1 + \mathbf{Z}\delta_2$, where $\dot{\Delta}$ is the root system of $\dot{\mathfrak{g}}$ and δ_1, δ_2 are two null roots corresponding to the affinizations. We define a *Chevalley basis* for $\mathfrak{t}$ in §2.1 from a Chevalley basis of $\dot{\mathfrak{g}}$ in the sense of Bourbaki [Bo2]. Then with this basis, in §2.2 we define Steinberg groups $St({}^{re}\Delta, R)$ for the root system ${}^{re}\Delta = \dot{\Delta} + \mathbf{Z}\delta_1 + \mathbf{Z}\delta_2$) over commutative rings by generators and relations (cf. Definition 2.2.3). It turns out that the usual Steinberg group $St(\dot{\Delta}, R_2)$ and $St({}^{re}\Delta, \mathbb{C})$ are isomorphic. From §2.3, we know the universal toroidal groups $\overline{G}$ satisfy the defining relations (R1′) and (R2′) of $St({}^{re}\Delta, \mathbb{C})$, hence there is a homomorphism ϕ from $St({}^{re}\Delta, \mathbb{C})$ onto $\overline{G}$. Also we conjecture that (R1′) and (R2′) together with another relation (R0′) are the defining relations of $\overline{G}$. In order to take a close look at $\overline{G}$, we review the vertex representations of $\mathfrak{t}$ (which are integrable) from [MRY] and [EMY] in §2.4. These representations provide us with a powerful tool with which to study the toroidal groups. In §2.5 we then are able to find out the kernel of $\phi\,|_{\widehat{N}}$ where $\widehat{N}$ is the subgroup of $St(\dot{\Delta}, R_2)$ defined in §2.2. By extending the root lattice spanned by Δ to a nondegenerate lattice, we may define a toroidal Weyl group which is then the semidirect product of the affine Weyl group and the affine root lattice (acting as translations). This generalizes the fact that the affine Weyl group is the semidirect product of the finite Weyl group and finite root lattice. We show that the finite, affine and toroidal Weyl groups are

isomorphic to the quotient group of $\widehat{N}$ mudulo some subgroups of $\widehat{N}$. In the same section we show that affine Kac-Moody groups can be imbedded into (universal) toroidal groups in two symmetric ways. With the help of algebraic K-theory and vertex representations, we prove in §2.6 our conjecture in the type $A_\ell(\ell \geq 4)$ case.

It is convenient to keep in mind that we will deal with Chevalley bases carefully. Here is the proceeding line. We fix a Chevalley basis for $\mathfrak{t}$ in §2.1. Relative to the chosen basis we define (exponential) elements $\tilde{x}_\mu(c)$ and $\bar{x}_\mu(c)$ in §2.2 and §2.3, and then in §2.4 we give Proposition 2.4.4 which makes the vertex representation V_Γ of $\mathfrak{t}$ into a toroidal group module ***explicitely*** and allows us to use calculations in [MRY] and [EMY] in Propositions 2.5.2 and 2.5.3.

Throughout this paper, the base field is the field $\mathbb{C}$ of complex numbers and the finite root system is of type A_ℓ, D_ℓ, or $E_\ell(\ell \geq 1)$ unless otherwise specified in the contents. For example, a module of a Lie algebra is first of all a complex vector space. Let A be a group. Then the group commutator $xyx^{-1}y^{-1}$ is denoted by (x, y). $Cent(A)$ stands for the center of A. If f is a homomorphism from A to another group B, the kernel of f is denoted by $ker(f)$.

§2.1. Toroidal Lie Algebras and Chevalley Bases.

In this section, we recall some basic concepts about toroidal Lie algebras. For further details the reader can consult [Sl] , [BM] and [MRY] . Also we present Chevalley bases for toroidal Lie algebras.

Let $R_2 = \mathbb{C}[s, s^{-1}, t, t^{-1}]$ be the commutative Laurant polynomial algebra of two variables over $\mathbb{C}$. Let $\dot{\mathfrak{g}}$ be a finite dimensional simple complex Lie algebra of type $X_\ell = A_\ell(\ell \geq 1), D_\ell(\ell \geq 4)$, or $E_\ell(\ell = 6, 7, 8)$. Then $R_2 \otimes \dot{\mathfrak{g}}$ is a Lie algebra by setting

$$[a \otimes x, b \otimes y] = ab \otimes [x, y]$$

for all $a, b \in R_2$ and $x, y \in \dot{\mathfrak{g}}$. Its *universal covering algebra* (uca) of $R_2 \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ (as a Lie algebra over $\mathbb{C}$) is called a **toroidal Lie algebra** (named in [MRY]) and is denoted by $\mathfrak{t}_{[2]}$ or simply $\mathfrak{t}$. We may describe $\mathfrak{t}$ explicitly as follows (see [Kas]):

Let Ω be the R_2-module with generators $d(s^p t^q)$ and relations $d(ab) = adb + bda$. Thus Ω is freely generated over R_2 by ds and dt and hence freely over $\mathbb{C}$ by $\{s^p t^q ds\} \cup \{s^p t^q dt\}$. dR_2 is the subspace over $\mathbb{C}$ generated by $d(s^p t^q)$. Then in the quotient space Ω/dR_2,

$$0 = \overline{d(s^p t^q)} = p\overline{s^{p-1} t^q ds} + q\overline{s^p t^{q-1} dt}.$$

Thus for $q \neq 0$

$$\overline{s^p t^{q-1} dt} = -\frac{p}{q}\,\overline{s^{p-1} t^q ds}$$

and a basis for Ω/dR_2 over $\mathbb{C}$ is

$$a(p, q) := \overline{s^{p-1} t^q ds}, \qquad (p, q) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$$

$$a(p, 0) := \overline{s^p t^{-1} dt}, \qquad p \in \mathbb{Z}$$

$$*(0, 0) := \overline{s^{-1} ds}.$$

Thus this basis is naturally parametrized by $(\mathbb{Z} \times \mathbb{Z}) \cup \{*\}$, where $\{*\}$ is just some singleton.

Let $\dot{\mathfrak{h}}$ be a Cartan subalgebra of $\dot{\mathfrak{g}}$. Let $\dot{\Delta} \subset \dot{\mathfrak{h}}^*$ denote the corresponding root system and let $\dot{\Pi} = \{\alpha_1, \cdots, \alpha_l\}$ be a base for $\dot{\Delta}$. The Killing form $(\cdot \mid \cdot)$ is non-degenerate on $\dot{\mathfrak{h}}$, and we will identify $\dot{\mathfrak{h}}^*$ with $\dot{\mathfrak{h}}$ by means of it. We assume that $(\cdot \mid \cdot)$ is so normalized that after this identification (long) roots in $\dot{\Delta}$ have square length equal to 2.

The complex vector space $\mathfrak{t} := R_2 \otimes \dot{\mathfrak{g}} \oplus \Omega/dR_2$ can be made into a Lie algebra by letting Ω/dR_2 be central, and setting

$$[a \otimes x, b \otimes y] = ab \otimes [x, y] + \overline{(da)b}(x \mid y) \tag{2.1.1}$$

for all $a, b \in R_2$ and $x, y \in \dot{\mathfrak{g}}$. Then $\mathfrak{t}$ is the uca of $R_2 \otimes \dot{\mathfrak{g}}$. The center is Ω/dR_2.

Note that $\mathfrak{t}$ has two subalgebras $\mathbb{C}[s, s^{-1}] \otimes \dot{\mathfrak{g}} + \mathbb{C}\overline{s^{-1}ds}$ and $\mathbb{C}[t, t^{-1}] \otimes \dot{\mathfrak{g}} + \mathbb{C}\overline{t^{-1}dt}$ which are isomorphic to the affine Lie algebra of type $X_\ell^{(1)}$. We denote the respective null roots by δ_1 and δ_2.

Let $\dot{Q}$ be the root lattice spanned by $\dot{\Delta}$. Then

$$\dot{\mathfrak{g}} = \bigoplus_{\alpha \in \dot{Q}} \dot{\mathfrak{g}}^\alpha$$

in which $\dot{\mathfrak{g}}^\alpha = \{x \in \dot{\mathfrak{g}} \mid [h, x] = \alpha(h)x, \forall h \in \dot{\mathfrak{h}} = \dot{\mathfrak{g}}^0\}$ and $\dot{\mathfrak{g}}^\alpha \neq (0)$ if and only if $\alpha \in \dot{\Delta} \cup \{0\}$.

We call the set

$$Q := \dot{Q} \oplus \mathbf{Z}\delta_1 \oplus \mathbf{Z}\delta_2$$

a **toroidal root lattice**. We extend the positive definite bilinear form $(\cdot \mid \cdot)$ on $\dot{Q}$ to a symmetric bilinear form on $Q \times Q$ by defining $(\alpha \mid \delta_1) = (\alpha \mid \delta_2) = 0$ for all $\alpha \in Q$. The new form $(\cdot \mid \cdot)$ is semi-positive definite with the radical $\mathbf{Z}\delta_1 \oplus \mathbf{Z}\delta_2$. We denote the radical by ${}^{im}\Delta$ and call its elements **imaginary roots**. Let

$${}^{re}\Delta := \{\mu = \alpha + m\delta_1 + n\delta_2 \mid \alpha \in \dot{\Delta}, m, n \in \mathbf{Z}\}.$$

Then ${}^{re}\Delta$ is the set of all elements $\alpha \in Q$ with $(\alpha \mid \alpha) = 2$. Elements in ${}^{re}\Delta$ are called **real roots**. Then

$$\Delta := \{\alpha + m\delta_1 + n\delta_2 \mid \alpha \in \dot{\Delta} \cup \{0\}, m, n \in \mathbf{Z}\}$$

is the disjoint union of ${}^{re}\Delta$ and ${}^{im}\Delta$ and is called a **toroidal root system** associated to $\dot{\Delta}$. The expression $\mu = \alpha + m\delta_1 + n\delta_2 \in Q$ will always mean the *unique expression* in the direct sum, so $\alpha \in \dot{Q}$.

Set for $\mu = \alpha + m\delta_1 + n\delta_2 \in Q\backslash\{0\}$,

$$\mathfrak{t}^\mu := s^m t^n \otimes \dot{\mathfrak{g}}^\alpha$$

and

$$\mathfrak{t}^0 := 1 \otimes \dot{\mathfrak{g}}^0 \oplus \Omega/dR_2.$$

Then $\mathfrak{t}^\mu \neq 0$ if and only if $\mu \in \Delta \cup \{0\}$ and

$$\mathfrak{t} = \bigoplus_{\mu \in Q} \mathfrak{t}^\mu.$$

But this does not define a Q-grading for $\mathfrak{t}$.

Since $\dot{\mathfrak{h}}^*$ and $\dot{\mathfrak{h}}$ are identified and $\dot{\mathfrak{g}}(\simeq 1 \otimes \dot{\mathfrak{g}})$ is a subalgebra of $\mathfrak{t}$, then we may identify Q as a subset of $\mathfrak{t}$ by setting

$$\delta_1 = \overline{s^{-1}ds}, \qquad \delta_2 = \overline{t^{-1}dt}.$$

This identification is justified by the choice of canonical basis of $\mathfrak{sl}_2(\mathbb{C})$ ($\mathfrak{sl}_2$-triplet).

Let $\dot{\varepsilon} : \dot{Q} \times \dot{Q} \to \{\pm 1\}$ be a bimultiplicative map satisfying

CC(i) $\quad \dot{\varepsilon}(\alpha, \alpha) = (-1)^{(\alpha|\alpha)/2}$

CC(ii) $\quad \dot{\varepsilon}(\alpha, \beta)\dot{\varepsilon}(\beta, \alpha) = (-1)^{(\alpha|\beta)}$

for all $\alpha, \beta \in \dot{Q}$. Such a cocycle on $\dot{Q}$ exists (see, for instance, Chapter 3 of [MP] or §2.1 of [Mit]).

In the sense of Bourbaki (Definition 3 of VIII §2 of [Bo2]), a Chevalley basis of $\dot{\mathfrak{g}}$ relative to (the Cartan subalgebra) $\dot{\mathfrak{h}}$ is a basis of form

$$\{E_\alpha(\in \dot{\mathfrak{g}}^\alpha) \mid \alpha \in \dot{\Delta}\} \cup \{\alpha_i \mid 1 \leq i \leq \ell\} \tag{2.1.2}$$

such that

(i) $[E_\alpha, E_{-\alpha}] = -\alpha$, for all $\alpha \in \dot{\Delta}$;

(ii) The linear map σ of $\dot{\mathfrak{g}}$ which is equal to -1 on $\dot{\mathfrak{h}}$ and maps E_α to $E_{-\alpha}$, is a (Lie algebra) automorphism of $\dot{\mathfrak{g}}$.

Then with respect to every cocycle $\dot{\varepsilon}$ on $\dot{Q}$ as above, according to Mitzman (§2.1 of [Mit]) $\dot{\mathfrak{g}}$ has a Chevalley basis of form (2.1.2) such that for all $\alpha, \beta \in \dot{\Delta}$,

$$[E_\alpha, E_\beta] = \begin{cases} 0, & \text{if } \alpha + \beta \notin \dot{\Delta} \cup \{0\} \\ -\alpha & \text{if } \alpha + \beta = 0 \\ \dot{\varepsilon}(\alpha, \beta)\, E_{\alpha+\beta} & \text{if } \alpha + \beta \in \dot{\Delta}, \end{cases}$$

EXAMPLE 2.1.1. *Suppose $\dot{\Delta}$ is of type A_ℓ. Then $\dot{\mathfrak{g}} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$. Then*

$$E_{ij}, -E_{ji}, H_i = E_{ii} - E_{i+1,i+1}$$

for $1 \le i < j \le \ell + 1$, is a Chevalley basis of $\dot{\mathfrak{g}}$ in the above sense (See Chapter VIII, §1 n° 1 of [Bo2]).

REMARK 2.1.2. *By a Chevalley basis of $\dot{\mathfrak{g}}$, we will mean that in the sense of Bourbaki. From now on, we fix a choice of a cocycle $\dot{\varepsilon}$ and a respective Chevalley basis as in (2.1.2), except that in §2.6, where we will use the chosen Chevalley basis of $\mathfrak{sl}_{\ell+1}(\mathbb{C})$ as in Example 2.1.1.*

We extend $\dot{\varepsilon}$ to a bimultiplicative map $\varepsilon : Q \times Q \to \{\pm 1\}$ by letting

CC(iii) $\quad \varepsilon(\alpha, \delta_1) = \varepsilon(\alpha, \delta_2) = 1$

for all $\alpha \in Q$. Let

$$\begin{aligned}
&\xi_\mu = s^m t^n \otimes E_\alpha, && \mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta \\
&\xi_i(m,n) = s^m t^n \otimes \alpha_i, && i = 1, \cdots, \ell; m, n \in \mathbb{Z} \\
&h_i = \xi_i(0,0) = 1 \otimes \alpha_i && i = 1, \cdots, \ell \\
&a(m,n) = \overline{s^{m-1} t^n ds}, && (m,n) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\}) \\
&a(m,0) = \overline{s^{m-1} t^{-1} dt}, && m \in \mathbb{Z}
\end{aligned}$$

$$\mathfrak{k}_t = \overline{t^{-1}dt}$$
$$\mathfrak{k}_s = \overline{s^{-1}ds}$$

DEFINITION 2.1.3. *A* **Chevalley basis** *of* $\mathfrak{t}$ *is a basis* $\mathcal{C}$ *of the form*

$$\{F_\mu (\in \mathfrak{t}^\mu) \mid \mu \in {}^{re}\Delta\} \cup \{\xi_i(m,n) \mid i = 1, \cdots, \ell; m, n \in \mathbf{Z}\}$$
$$\cup \{a(m,n) \mid m, n \in \mathbf{Z}\} \cup \{\mathfrak{k}_s\}$$

such that

(i) $[F_\mu, F_{-\mu}] = -\mu$, *for all* $\mu \in {}^{re}\Delta$.

(ii) the linear map $\sigma : \mathfrak{t} \to \mathfrak{t}$ *defined by*

$$\sigma(F_\mu) := F_{-\mu}$$
$$\sigma(\xi_i(m,n)) := -\xi_i(-m,-n)$$
$$\sigma\,|_{\Omega/dR_2} := -1$$

for all $\mu \in {}^{re}\Delta, 1 \le i \le \ell$, *and* $m, n \in \mathbf{Z}$, *is a Lie algebra automorphism of* $\mathfrak{t}$. *We call it the* **Chevalley involution** *of* $\mathfrak{t}$ *relative to* $\mathcal{C}$.

PROPOSITION 2.1.4.

$$\mathcal{B} := \{\xi_\mu \mid \mu \in {}^{re}\Delta\} \cup \{\xi_i(m,n) \mid 1 \le i \le \ell, m, n \in \mathbf{Z}\} \cup \{a(m,n) \mid m, n \in \mathbf{Z}\} \cup \{\mathfrak{k}_s, \mathfrak{k}_t\}$$

is a Chevalley basis of $\mathfrak{t}$ *with respect to the chosen Chevalley basis of* $\dot{\mathfrak{g}}$. *Moreover,*

$$[\xi_\mu, \xi_\nu] = \begin{cases} 0 & \text{if } \mu + \nu \notin \Delta \\ -\mu & \text{if } \mu + \nu = 0 \\ \varepsilon(\mu,\nu)\xi_{\mu+\nu} & \text{if } \mu + \nu \in {}^{re}\Delta, \end{cases}$$

for all $\mu, \nu \in {}^{re}\Delta$ *and the structure constants relative to* $\mathcal{B}$ *are all integers.* □

PROOF: From defintions of elements in $\mathcal{B}$ we see that $\mathcal{B}$ is a basis of $\mathfrak{t}$. Let $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$. First we note that for each $\alpha \in \dot{\Delta}$, $(E_\alpha \mid E_{-\alpha}) = -1$. This follows from the $\dot{\mathfrak{g}}$-invariance of the Killing form $(\cdot \mid \cdot)$ on $\dot{\mathfrak{g}}$. Indeed,

$$2(E_\alpha \mid E_{-\alpha}) = ([\alpha, E_\alpha] \mid E_{-\alpha}) = (\alpha \mid [E_\alpha, E_{-\alpha}]) = (\alpha \mid -\alpha) = -2.$$

Then by (2.1.1) and the identification $Q \subset \mathfrak{t}$,

$$[\xi_\mu, \xi_{-\mu}] = 1 \otimes [E_\alpha, E_{-\alpha}] + (m\overline{s^{-1}ds} + n\overline{t^{-1}dt})(E_\alpha \mid E_{-\alpha}) = -\alpha - m\mathfrak{c}_s - n\mathfrak{c}_t = -\mu.$$

So $\mathcal{B}$ satisfies condition(i) in Definition 2.1.3. By calculating other possible brackts between any pair of elements in $\mathcal{B}$, we see that the structure constants are integers and in the meantime the linear map σ defined as in Definition 2.1.3 is an automorphism of $\mathfrak{t}$. Thus the proposition follows. □

We conclude this section by showing that the structure constants of any Chevalley basis of $\mathfrak{t}$ are integers. We proceed this as in §2.3 of [Mit].

Denote by the linear projection from $Q \to \dot{Q}$ defined by

$$\overline{\alpha + m\delta_1 + n\delta_2} := \alpha. \tag{2.1.3}$$

Then $\overline{re\Delta} = \dot{\Delta}$ and $\overline{im\Delta} = 0$.

We need to extend the normalized Killing form $(\cdot \mid \cdot)$ on $\dot{\mathfrak{g}}$ to a symmetric, $\mathfrak{t}$-invariant bilinear form, also denoted by $(\cdot \mid \cdot)$, on $\mathfrak{t}$ by making the center Ω/R_2 of $\mathfrak{t}$ lie in the radical of $(\cdot \mid \cdot)$ and setting

$$(s^m t^n \otimes x \mid s^p t^q \otimes y) = \delta_{m+p,0}\delta_{n+q,0}(x \mid y)$$

where $x, y \in \dot{\mathfrak{g}}$, and $m, n \in \mathbf{Z}$. The $\mathfrak{t}$-invariance of this form follows from that $(\cdot \mid \cdot)$ is $\dot{\mathfrak{g}}$-invariant on $\dot{\mathfrak{g}}$, for example,

$$\begin{aligned}
&([s^m t^n \otimes x, s^p t^q \otimes y] \mid s^i t^j \otimes z) \\
&\quad = (s^{m+p} t^{n+q} \otimes [x, y] \mid s^i t^j \otimes z) \\
&\quad = \delta_{m+p+i,0}\delta_{n+q+j,0}([x, y] \mid z) \\
&\quad = \delta_{m+p+i,0}\delta_{n+q+j,0}(x \mid [y, z]) \\
&\quad = (s^m t^n \otimes x \mid [s^p t^q \otimes y, s^i t^j \otimes z]).
\end{aligned}$$

Now let $\mathcal{C}$ be a basis of $\mathfrak{t}$ having the form of a Chevalley basis of $\mathfrak{t}$ as given by Definition 2.1.3, but do not assume that $\mathcal{C}$ satisfies conditions (i) and (ii) of Definition 2.1.3.

For $\alpha \in \dot{\Delta}$ and $\nu = m\delta_1 + n\delta_2 \in {}^{im}\Delta$, define

$$\xi_\alpha(\nu) := \xi_\alpha(m\delta_1 + n\delta_2) := s^m t^n \otimes \alpha. \tag{2.1.4}$$

Note that

$$\xi_\alpha(\nu) = -\xi_{-\alpha}(\nu). \tag{2.1.5}$$

For $\mu, \nu \in \Delta$, define the constants $N_{\mu,\nu} \in \mathbb{C}$ as follows:

$$[F_\mu, F_\nu] = N_{\mu,\nu} F_{\mu+\nu} \qquad \text{if } \mu, \nu \in {}^{re}\Delta, \mu + \nu \in {}^{re}\Delta, \tag{2.1.6a}$$

$$[F_\mu, F_\nu] = N_{\mu,\nu}\xi_{\bar{\mu}}(\mu + \nu) + z, \text{for some } z \in \Omega/R_2, \tag{2.1.6b}$$

$$\text{if } \mu, \nu \in {}^{re}\Delta, \mu + \nu \in {}^{im}\Delta,$$

$$[F_\mu, \xi_{\bar{\mu}}(\nu)] = N_{\mu,\nu} F_{\mu+\nu} \qquad \text{if } \mu \in {}^{re}\Delta, \nu \in {}^{im}\Delta, \tag{2.1.6c}$$

$$[\xi_{\bar{\mu}}(\nu), F_\mu] = N_{\nu,\mu} F_{\mu+\nu} \qquad \text{if } \mu {}^{re}\Delta, \nu \in {}^{im}\Delta, \tag{2.1.6d}$$

$$N_{\mu,\nu} = -\delta_{\mu+\nu,0} \qquad \text{if } \mu, \nu \in {}^{im}\Delta, \tag{2.1.6e}$$

$$N_{\mu,\nu} = 0 \qquad \text{if } \mu + \nu \in \Delta. \tag{2.1.6f}$$

Then for $\mu, \nu \in \Delta$, by using the Lie multiplication of $\mathfrak{t}$, the skew-symmetry of the Lie bracket and (2.1.5), we have

$$N_{\mu,\nu} = -N_{\nu,\mu} \qquad \text{if } \overline{\mu + \nu} \neq 0, \tag{2.1.7a}$$

$$N_{\mu,\nu} = N_{\nu,\mu} \qquad \text{if } \overline{\mu + \nu} = 0, \tag{2.1.7b}$$

$$N_{\mu,\nu} = -N_{\nu,\mu} = -2 \qquad \text{if } \mu \in {}^{re}\Delta, \nu \in {}^{im}\Delta, \tag{2.1.7c}$$

$$N_{\mu,\nu} = 0 \text{ if and only if } N_{\nu,\mu} = 0. \tag{2.1.7d}$$

The conditions (i) and (ii) of Definition 2.1.3 imply that

$$N_{\mu,-\mu} = -1 \qquad \mu \in {}^{re}\Delta, \tag{2.1.8a}$$

$$N_{\mu,\nu} = N_{-\mu,-\nu} \qquad \text{for all } \mu, \nu \in \Delta, \tag{2.1.8b}$$

respectively.

Suppose $\mu \in {}^{re}\Delta, \nu \in \Delta$. Let

$$S(\nu, \mu) := \{\nu + k\mu \mid k \in \mathbf{Z}\} \cap \Delta.$$

$S(\nu, \mu)$ is called the μ-root string through ν.

LEMMA 2.1.5. *Suppose $\mu \in {}^{re}\Delta, \nu \in \Delta$. There exist non-negative integers d, u such that $d - u = (\mu \mid \nu)$ and*

$$S(\nu, \mu) = \{\nu + k\mu \mid -d \leq k \leq u\}.$$

Explicitly,

$$S(\nu, \mu) = \begin{cases} \{\nu - 2\mu, \nu - \mu, \nu\} & \text{if } (\mu \mid \nu) = 2 \\ \{\nu, \nu + \mu, \nu + 2\mu\} & \text{if } (\mu \mid \nu) = -2 \\ \{\nu - \mu, \nu\} & \text{if } (\mu \mid \nu) = 1 \\ \{\nu, \nu + \mu\} & \text{if } (\mu \mid \nu) = -1 \\ \{\nu\} & \text{if } (\mu \mid \nu) = 0 \text{ and } \nu \in {}^{re}\Delta, \\ \{\nu - \mu, \nu, \nu + \mu\} & \text{if } (\mu \mid \nu) = 0 \text{ and } \nu \in {}^{im}\Delta, \end{cases}$$

□

The proof for this lemma is easy. One notes that the value $(\mu \mid \nu)$ almost determines how ν behaves, for example, $(\mu \mid \nu) = -2$ implies $\nu \in {}^{re}\Delta$ and $\bar{\nu} = -\bar{\mu}$. The six cases given in the lemma exhaust all possibilities.

PROPOSITION 2.1.6. *Suppose $\mu \in {}^{re}\Delta, \nu \in \Delta$. Let d, u be integers determined as in Lemma 2.1.5. If $N_{\mu,\nu} \neq 0$, then*

$$N_{\mu,\nu} N_{-\mu,-\nu} = -(N_{\mu,-\mu} N_{\nu,-\nu} / N_{\mu+\nu,-\mu-\nu})(d+1)^2.$$

PROOF: Since $N_{\mu,\nu} \neq 0$, then $\mu + \nu \in \Delta$. So we have three cases to consider, which are

Case 1: $(\mu \mid \nu) = -1$ i.e., $\nu, \mu + \nu \in {}^{re}\Delta$,

Case 2: $(\mu \mid \nu) - 2$ i.e., $\mu + \nu \in {}^{im}\Delta$,

Case 3: $\nu \in {}^{im}\Delta$.

With respect to these three cases, the pair $(d, u) = (0,1), (-1,1), (0,2)$, respectively. First we have the following formulas:

$$N_{\mu,\nu}N_{-\mu,\mu+\nu} = \begin{cases} N_{\mu,-\mu}u(d+1) & \text{in Case 1} \\ -N_{\mu,-\mu}u(d+1) & \text{in Case 2 or 3.} \end{cases} \tag{2.1.9}$$

For the following three (Jacobi) identities together with definitions of $N_{\mu,\nu}$ give us the respective formulas.

$$[F_{-\mu},[F_\mu,F_\nu]] = [[F_{-\mu},F_\mu],F_\nu] + [F_\mu,[F_{-\mu},F_\nu]] = [[F_{-\mu},F_\mu],F_\nu],$$

$$[F_{-\mu},[F_\mu,F_\nu]] = [[F_{-\mu},F_\mu],F_\nu] + [F_\mu,[F_{-\mu},F_\nu]] = [[F_{-\mu},F_\mu],F_\nu],$$

$$[F_\mu,[F_{-\mu},F_{\mu+\nu}]] = [[F_\mu,F_{-\mu}],F_{\mu+\nu}] + [F_{-\mu},[F_\mu,F_{\mu+\nu}]] = [[F_{-\mu},F_\mu],F_\nu].$$

Using t-invariance of $(\cdot \mid \cdot)$, we have for all $\lambda \in {}^{re}\Delta$ that

$$2(F_\lambda \mid F_{-\lambda}) = ([\lambda,F_\lambda] \mid F_{-\lambda}) = (\lambda \mid [F_\lambda,F_{-\lambda}]) = N_{\lambda,-\lambda}(\lambda \mid \lambda).$$

So

$$(F_\lambda \mid F_{-\lambda}) = N_{\lambda,-\lambda}. \tag{2.1.10}$$

Now we show the proposition case by case.

Assume $\nu, \mu+\nu \in {}^{re}\Delta$. Then

$$\begin{aligned} N_{-\mu,\mu+\nu}(F_\nu \mid F_{-\nu}) &= ([F_{-\mu},F_{\mu+\nu}] \mid F_{-\nu}) \\ &= (-F_{\mu+\nu} \mid [F_{-\mu},F_{-\nu}]) = -N_{-\mu,-\nu}(F_{\mu+\nu} \mid F_{-\mu-\nu}). \end{aligned}$$

Thus by (2.1.10),

$$N_{-\mu,\mu+\nu}N_{\nu,-\nu} = -N_{-\mu,-\nu}N_{\mu+\nu,-\mu-\nu}. \tag{2.1.11}$$

From (2.1.9) and (2.1.11), we get

$$N_{\mu,\nu}N_{-\mu,-\nu}N_{\mu+\nu,-\mu-\nu} = -N_{\mu,-\mu}N_{\nu,-\nu}.$$

This is the equation in the proposition in Case 1, since $(d,u) = (0,1)$ and $(d+1)^2 = 1$.

Assume $\mu + \nu \in {}^{im}\Delta$. Let $\alpha = \bar{\mu} = -\bar{\nu}$. Then

$$N_{-\mu,\mu+\nu}(F_\nu \mid F_{-\nu}) = ([F_{-\mu}, \xi_{-\alpha}(\mu+\nu)] \mid F_{-\nu}) = -(\xi_{-\alpha}(\mu+\nu) \mid [F_{-\mu}, F_{-\nu}])$$
$$= -N_{-\mu,-\nu}(\xi_{-\alpha}(\mu+\nu) \mid \xi_{-\alpha}(-\mu-\nu)) = -2N_{-\mu,-\nu}.$$

Thus

$$N_{-\mu,\mu+\nu}N_{\nu,-\nu} = -2N_{-\mu,-\nu}. \tag{2.1.12}$$

Also (2.1.9) and (2.1.12) imply the proposition in Case 2.

Finally assume $\nu \in {}^{im}\Delta$ and $\alpha = \bar{\mu}$. From (2.1.12), we have

$$N_{-\mu,\mu+\nu}(\xi_{-\alpha}(\nu) \mid \xi_{-\alpha}(-\nu)) = -N_{-\mu,-\nu}(F_{\mu+\nu} \mid F_{-\mu-\nu}).$$

Thus by (2.1.10),

$$2N_{-\mu,\mu+\nu} = -N_{-\mu,-\nu}N_{\mu+\nu,-\mu-\nu}.$$

Again using (2.1.9), we get the proposition in Case 3. □

An immediate consequence of Proposition 2.1.6 and (2.1.8) is

COROLLARY 2.1.7. *Suppose that μ and ν as in Proposition 2.1.6. If $\mathcal{C}$ satisfies condition(i) of Definition 2.1.3, then*

$$N_{\mu,\nu}N_{-\mu,-\nu} = (d+1)^2.$$

If in addition condition(ii) of Definition 2.1.3 holds for $\mathcal{C}$, that is, $\mathcal{C}$ is a Chevalley basis of $\mathfrak{t}$, *then*

$$N_{\mu,\nu} = \pm(d+1).$$

□

Let $\mathcal{C}$ be a Chevalley basis of $\mathfrak{t}$ as given in Definition 2.1.3. We write the elements of $\mathcal{C}$ in terms of the Chevalley basis $\mathcal{B}$ given in Proposition 2.1.4. For $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$, we may write

$$F_\mu = c_\mu s^m t^n \otimes E_\alpha = c_\mu \xi_\mu \tag{2.1.13}$$

for some $c_\mu \in \mathbb{C}^\times$.

Let $\mu = \alpha + m\delta_1 + n\delta_2, \nu = p\delta_1 + q\delta_2 \in {}^{re}\Delta$ and $\beta \in \dot{\Delta}$. Then

$$\begin{aligned}[\xi_\beta(\nu), F_\mu] &= [s^p t^q \otimes \beta, c_\mu s^m t^n \otimes E_\alpha] \\ &= (\beta \mid \alpha) c_\mu s^{m+p} t^{n+q} \otimes E_\alpha \\ &= (\beta \mid \alpha)(c_\mu / c_{\mu+\nu}) F_{\mu+\nu}.\end{aligned} \tag{2.1.14}$$

In particular, when $\beta = \alpha$, we have

$$N_{\nu,\mu} = 2c_\mu / c_{\mu+\nu}. \tag{2.1.15}$$

By Corollary 2.1.7, $N_{\nu,\mu} = \pm 2$. So (2.1.15) gives

$$c_\mu = \pm c_{\mu+\nu} \tag{2.1.16}$$

for all $\mu \in {}^{re}\Delta$ and $\nu \in {}^{im}\Delta$.

Also Corollary 2.1.7, (2.1.6a, b) and (2.1.13) give

$$c_\mu c_\nu = \pm 1 \tag{2.1.17}$$

for all $\mu, \nu \in {}^{re}\Delta$ with $\mu + \nu \in \Delta$.

Now we can prove our result.

PROPOSITION 2.1.8. *Let $\mathcal{C}$ be a Chevalley basis of $\mathfrak{t}$ as given in Definition 2.1.3. Then the structure contants relative to $\mathcal{C}$ are all integers.*

PROOF: We need to show that for any two elements $X, Y \in \mathcal{C}$, $[X, Y]$ is an integral linear combinition of elements in $\mathcal{C}$.

Suppose $X = F_\mu, Y = F_\nu$ for some $\mu, \nu \in {}^{re}\Delta$. Because of Corollary 2.1.7 and (2.1.6a), it suffices to check the case $\mu + \nu \in {}^{im}\Delta$. Using (2.1.13) and (2.1.17) together with the Lie multiplication (2.1.1), we get the result in this case.

If $X, Y \in \{\xi_i(m,n) \mid i = 1, \cdots, \ell, m, n \in \mathbb{Z}\}$, our result holds by Proposition 2.1.4.

The only remaining case is that $X = F_\mu, Y = \xi_i(m,n)$ for some $\mu \in {}^{re}\Delta, 1 \leq i \leq \ell, m, n \in \mathbf{Z}$. But $[X,Y]$ is again an integral combinition of elements of $\mathcal{C}$ because of (2.1.16) and (2.1.14). This completes the proof. □

Also, we have

PROPOSITION 2.1.9. *Suppose $\mathcal{C}$ is a Chevalley basis of* t *as given in Definition 2.1.3. For each $\alpha \in \dot{\Delta}$, define E'_α by*

$$F_\alpha = 1 \otimes E'_\alpha. \tag{2.1.18}$$

Then

$$\{E'_\alpha \mid \alpha \in \dot{\Delta}\} \cup \{\alpha_i \mid 1 \leq i \leq \ell\}$$

is a Chevalley basis of $\dot{\mathfrak{g}}$ and

$$F_\mu = \pm s^m t^n \otimes E'_\alpha, \tag{2.1.19}$$

for each $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$.

PROOF: Indeed, $\dot{\mathfrak{g}}$ is a subalgebra of t under the canonical injection $X \mapsto 1 \otimes X$, for $X \in \dot{\mathfrak{g}}$. The restriction of the Chevalley involution of t (relative to $\mathcal{C}$) gives the Chevalley involution of $\dot{\mathfrak{g}}$. Then it suffices to show (2.1.19). By Corollary 2.1.7, $N_{\alpha, m\delta_1 + n\delta_2} = \pm 2$. Then by (2.1.6c), we have

$$\pm 2F_\mu = N_{\alpha, m\delta_1 + n\delta_2} F_\mu = [F_\alpha, s^m t^n \otimes \alpha] = -2s^m t^n \otimes E'_\alpha.$$

So (2.1.19) holds. □

§2.2. Steinberg Groups $St(\dot{\Delta}, R)$ and $St({}^{re}\Delta, R)$.

Steinberg introduced Steinberg groups for finite root systems in [St1]. This notion has been generalized for root systems of Kac-Moody algebras (see, for example, [Ti1]). In this section, we recall some concepts about Steinberg groups for finite root systems $\dot{\Delta} = A_\ell, D_\ell$, and E_ℓ and define Steinberg groups for (real) toroidal root system of these types. Most results in this section are standard and for completeness we exihibit some proofs.

DEFINITION 2.2.1. *Assume R is a commutative and associative ring with the identity 1. The* **Steinberg group** *$St(\dot{\Delta}, R)$ is the abstract group with the following presentation:*

generators : $\hat{x}_\alpha(a); \alpha \in \dot{\Delta}, a \in R;$

relations:

(R1) $\hat{x}_\alpha(a)\hat{x}_\beta(b) = \hat{x}_\alpha(a+b)$

(R2) $\left(\hat{x}_\alpha(a), \hat{x}_\beta(b)\right) = \begin{cases} 1, & \text{if } \alpha+\beta \notin \dot{\Delta} \cup \{0\} \\ \hat{x}_{\alpha+\beta}(\dot{\epsilon}(\alpha,\beta)ab), & \text{if } (\alpha \mid \beta) = -1\ , \end{cases}$

where $\dot{\epsilon}(\alpha, \beta)$ is fixed in Remark 2.1.2.

If $\operatorname{rank} \dot{\Delta} = 1$, the relation (R2) is vacuous and is replaced by (R3) below. $St(\dot{\Delta}, R)$ is independent of a choice of a cocycle or a Chevalley basis, up to isomorphism. $St(\dot{\Delta}, R)$ is functorial in R. Some references for this subject are [HO], [Kal], [Ma], [Mi], [Ste], [St1&2].

For $\alpha \in \dot{\Delta}, u \in R^\times$(the units group of R), define $\hat{n}_\alpha(u), \hat{h}_\alpha(u) \in St(\dot{\Delta}, R)$ by the formulas

$$\hat{n}_\alpha(u) := \hat{x}_\alpha(u)\hat{x}_{-\alpha}(u^{-1})\hat{x}_\alpha(u)$$

$$\hat{h}_\alpha(u) := \hat{n}_\alpha(u)\hat{n}_\alpha(-1),$$

and let $\widehat{N}(\dot{\Delta}, R)$ (respectivly, $\widehat{H}(\dot{\Delta}, R)$) be the subgroup generated by all $\hat{n}_\alpha(u)$ (respectively, $\hat{h}_\alpha(u)$), $\alpha \in \dot{\Delta}, u \in R^\times$. Then for $\alpha, \beta \in \dot{\Delta}, a, b \in R, u, v \in R^\times$, the

following relations are consequences of (R1) and (R2) if rank $\dot{\Delta} \geq 2$, and of (R1) and (R3) if rank $\dot{\Delta} = 1$.

(R3) $\hat{n}_\alpha(u)\hat{x}_\beta(a)\hat{n}_\alpha(u)^{-1} = \hat{x}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)}a)$

(R4) $\hat{n}_\alpha(u)\hat{n}_\beta(v)\hat{n}_\alpha(u)^{-1} = \hat{n}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)}v)$

$\hat{n}_\alpha(u) = \hat{n}_{-\alpha}(u^{-1})$

(R5) $\hat{n}_\alpha(u)\hat{h}_\beta(v)\hat{n}_\alpha(u)^{-1} = \hat{h}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)}v)\hat{h}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)})^{-1}$

(R6) $\hat{h}_\alpha(u)\hat{x}_\beta(a)\hat{h}_\alpha(u)^{-1} = \hat{x}_\beta(u^{(\beta|\alpha)}a)$

(R7) $\hat{h}_\alpha(u)\hat{n}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{n}_\beta(u^{(\beta|\alpha)}v)$

$\hat{h}_\alpha(u) = \hat{h}_{-\alpha}(u^{-1})$

(R8) $\hat{h}_\alpha(u)\hat{h}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{h}_\beta(u^{(\beta|\alpha)}v)\hat{h}_\beta(u^{(\beta|\alpha)})^{-1}$

where if $(\alpha \mid \beta) = \pm 1$, η is equal to $\dot{\varepsilon}(\alpha, \beta)$, otherwise, η is equal to 1.

Warning: These relations may be presented in a different appearance as in the references mentioned above because of the different meaning of Chevalley bases.

PROOF OF (R3)-(R8): Let us assume (R3) first. Then the first formula of (R4) is immediate from (R3) and the definition $\hat{n}_\alpha(u)$. Letting $\beta = \alpha, v = u$ in the first formula of (R4), we get the second formula in (R4) since $\eta(\alpha, \alpha) = 1$. And hence the second formula of (R7) holds. Applying (R4) to $\hat{h}_\alpha(v) = \hat{n}_\alpha(v)\hat{n}_\alpha(-1)$, we get (R5). Twice applications of (R3) give (R6). Then (R6) implies the first formula of (R7), then (R8). It remains to prove (R3) in the cases of rank $\dot{\Delta} \geq 2$.

If $(\alpha \mid \beta) = 0$, then (R3) holds since $\eta(\alpha, \beta) = 1$ in this case.

Assume $(\alpha \mid \beta) = -1$, that is, $\alpha + \beta \in \dot{\Delta}$. By using $\dot{\varepsilon}(-\alpha, \alpha + \beta)\dot{\varepsilon}(\alpha, \beta) = -1$, $(\alpha + \beta \mid \alpha) \geq 0$ and (R1)-(R2), we have

$\hat{n}_\alpha(u)\hat{x}_\beta(a)\hat{n}_\alpha(u)^{-1}$

$$= \hat{x}_\alpha(u)\hat{x}_{-\alpha}(u^{-1})\hat{x}_\alpha(u)\hat{x}_\beta(a)\hat{x}_\alpha(-u)\hat{x}_{-\alpha}(-u^{-1})\hat{x}_\alpha(-u)$$

$$= \hat{x}_\alpha(u)\hat{x}_{-\alpha}(u^{-1})\hat{x}_{\alpha+\beta}(\dot{\varepsilon}(\alpha,\beta)ua)\hat{x}_\beta(a)\hat{x}_{-\alpha}(-u^{-1})\hat{x}_\alpha(-u)$$

$$= \hat{x}_\alpha(u)\hat{x}_{\alpha+\beta}(\dot{\varepsilon}(\alpha,\beta)ua)\hat{x}_\beta(\dot{\varepsilon}(-\alpha,\alpha+\beta)\dot{\varepsilon}(\alpha,\beta)a)\hat{x}_\beta(a)\hat{x}_\alpha(-u)$$

$$= \hat{x}_{\alpha+\beta}(\dot{\varepsilon}(\alpha,\beta)ua)$$

$$= \hat{x}_{r_\alpha\beta}(\eta(\alpha,\beta)ua),$$

which is (R3). When $(\alpha \mid \beta) = 1$, the same proof applies.

Since $\operatorname{rank}\dot{\Delta} \geq 2$, for each $\alpha \in \dot{\Delta}$ there are $\beta, \gamma \in \dot{\Delta}$ such that $\beta + \gamma = \alpha$. Then $(\beta \mid \gamma) = -1$ and $(\alpha \mid \beta) = 1 = (\alpha \mid \gamma)$. By above, we know (R3) holds for any two roots with their inner product value ± 1. Then

$$\hat{n}_\alpha(u)\hat{x}_\alpha(a)\hat{n}_\alpha(u)^{-1}$$

$$= \hat{n}_\alpha(u)\big(\hat{x}_\beta(\dot{\varepsilon}(\beta,\gamma)a), \hat{x}_\gamma(1)\big)\hat{n}_\alpha(u)^{-1}$$

$$= \big(\hat{n}_\alpha(u)\hat{x}_\beta(\dot{\varepsilon}(\beta,\gamma)a)\hat{n}_\alpha(u)^{-1}, \hat{n}_\alpha(u)\hat{x}_\gamma(1)\hat{n}_\alpha(u)^{-1}\big)$$

$$= \big(\hat{x}_{r_\alpha\beta}(\dot{\varepsilon}(\alpha,\beta)u^{-(\alpha|\beta)}\dot{\varepsilon}(\beta,\gamma)a), \hat{x}_{r_\alpha\gamma}(\dot{\varepsilon}(\alpha,\gamma)u^{-(\alpha|\gamma)})\big)$$

$$= \hat{x}_{r_\alpha(\beta+\gamma)}\big(\dot{\varepsilon}(r_\alpha\beta, r_\alpha\gamma)\dot{\varepsilon}(\alpha,\beta)\dot{\varepsilon}(\beta,\gamma)\dot{\varepsilon}(\alpha,\gamma)u^{-2}a\big)$$

$$= \hat{x}_{-\alpha}(u^{-2}a).$$

Similarly, we have

$$\hat{n}_\alpha(u)\hat{x}_{-\alpha}(a)\hat{n}_\alpha(u)^{-1} = \hat{x}_\alpha(u^2a).$$

These two equations complete the proof of (R3). □

For $u, v \in R^\times, \alpha \in \dot{\Delta}$, the element

$$\{u, v\}_\alpha := \hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1}\hat{h}_\alpha(v)^{-1}$$

is called a **Steinberg symbol**. The subgroup generated by all the Steinberg symbols is denoted by $\widehat{C}(\dot{\Delta}, R)$. The following lemma is standard (see §1.5 of [HO], §3.2 of [Kal], §5 of [Ma] and §8& 9 of [Mi]). We give a part of its proof.

LEMMA 2.2.2.

(i) $\widehat{C}(\dot{\Delta}, R)$ *is contained in the center of* $St(\dot{\Delta}, R)$.

(ii) If $\operatorname{rank}\dot{\Delta} \geq 2$, *then* $\{u, v\hat{\}}_\alpha$ *is independent of the choices of* α. *In this case, we write*

$$\{u, v\hat{\} } := \{u, v\hat{\}}_\alpha$$

and then $\{u, v\hat{\}}$ *satisfies:*

$$\textit{skew-symmetry} \quad \{u, v\hat{\} } = \{v, u\hat{\}}^{-1} \tag{2.2.1}$$

$$\textit{bi-multiplicativity:} \quad \{u, vw\hat{\} } = \{u, v\hat{\}}\{u, w\hat{\} } \tag{2.2.2}$$

$$\{uv, w\hat{\} } = \{u, w\hat{\}}\{v, w\hat{\} }$$

$$\{u, -u\hat{\} } = 1 \tag{2.2.3}$$

$$\{u, 1-u\hat{\} } = 1 \quad \textit{if } 1-u \in R^\times \tag{2.2.4}$$

PROOF: We imitate proofs in §1.5 of [HO]. (i) It suffices to show that $\{u, v\hat{\}}_\alpha$ commute with $\hat{x}_\beta(a)$ for any given $\beta \in \dot{\Delta}$ and $a \in R$. But this follows from (R6).

(ii) Suppose $\alpha, \beta \in \dot{\Delta}$ with $\alpha + \beta \in \dot{\Delta}$. We show that

$$\{u, v\hat{\}}_\alpha = \{u, v\hat{\}}_\beta. \tag{2.2.5}$$

This implies that $\{u, v\hat{\}}_\alpha$ does not depend on the choice of α since any two roots α, β in $\dot{\Delta}$ can be *jointed* by roots $\beta_1 = \alpha, \beta_2, \cdots, \beta_k = \beta$ with $(\beta_i \mid \beta_{i+1}) = -1$.

The following calculation uses (i) and (R5).

$$\begin{aligned}
\{u, v\hat{\}}_\beta &= \hat{n}_\alpha(\eta)\{u, v\hat{\}}_\beta \hat{n}_\alpha(-\eta) \\
&= \hat{n}_\alpha(\eta)\hat{h}_\beta(uv)\hat{h}_\beta(u)^{-1}\hat{h}_\beta(v)^{-1}\hat{n}_\alpha(-\eta) \\
&= \hat{h}_{r_\alpha\beta}(uv)\hat{h}_{r_\alpha\beta}(1)^{-1}\hat{h}_{r_\alpha\beta}(1)\hat{h}_{r_\alpha\beta}(u)^{-1}\hat{h}_{r_\alpha\beta}(1)\hat{h}_{r_\alpha\beta}(v)^{-1} \\
&= \{u, v\hat{\}}_{r_\alpha\beta}
\end{aligned}$$

where $\eta = \dot{\epsilon}(\alpha, \beta)$. Now it is clear that (2.2.5) holds. The rest of proof will be omitted. □

Lemma 2.2.2(ii) is not necessarily true in the case $\dot{\Delta} = A_1$. See remarks in §1.5A (Page 63) of [HO].

Suppose $S \subset \dot{\Delta}$. Let $\widehat{U}(S, R)$ be the subgroup of $St(\dot{\Delta}, R)$ generated by all $\hat{x}_\alpha(a), \alpha \in S, a \in R$. We will often use the abbreviations

$$\widehat{U}^\alpha := \widehat{U}(\alpha, R)$$
$$\widehat{U}^\pm := \widehat{U}(\dot{\Delta}_\pm, R)$$

For any toroidal Lie algebra $\mathfrak{t}$, we may choose a Chevalley basis (Definition 2.1.3) and then define Steinberg groups for the toroidal root system of $\mathfrak{t}$ over commutative rings. Specifically, we will use the Chevalley basis $\mathcal{B}$ of $\mathfrak{t}$ in Proposition 2.1.4. with the cocycle $\varepsilon(\mu, \nu)$ on $Q \times Q$ associated to $\dot{\epsilon}(\alpha, \beta)$ for the purpose of using representation theory.

DEFINITION 2.2.3. *Assume R is a commutative and associative ring with the identity 1. Let Δ be a toroidal root system defined in §2.1. The* **Steinberg group** *$St({}^{re}\Delta, R)$ is the abstract group with the following presentation:*

generators : $\tilde{x}_\mu(a); \quad \mu \in {}^{re}\Delta, a \in R$

relations:

(R1′) $\tilde{x}_\mu(a)\tilde{x}_\mu(b) = \tilde{x}_\mu(a+b)$

(R2′) $(\tilde{x}_\mu(a), \tilde{x}_\nu(b)) = \begin{cases} 1, & \text{if } \mu + \nu \notin {}^{re}\Delta \cup \{0\} \\ \tilde{x}_{\mu+\nu}(\varepsilon(\mu,\nu)ab), & \text{if } (\mu \mid \nu) = -1, \text{ i.e., } \mu + \nu \in {}^{re}\Delta\ . \end{cases}$

If $\operatorname{rank} \dot{\Delta} = 1$, relation (R2′) is vacuous and is replaced by (R3′) below. $St({}^{re}\Delta, R)$ is functorial in R.

For $\mu \in {}^{re}\Delta, u \in R^\times$(the units group of R), define $\tilde{n}_\mu(u), \tilde{h}_\mu(u) \in St({}^{re}\Delta, R)$ by the formulas

$$\tilde{n}_\mu(u) := \tilde{x}_\mu(u)\tilde{x}_{-\mu}(u^{-1})\tilde{x}_\mu(u)$$

$$\tilde{h}_\mu(u) := \tilde{n}_\mu(u)\tilde{n}_\mu(-1),$$

and let $\widetilde{N}({}^{re}\Delta, R)$ (respectively, $\widetilde{H}({}^{re}\Delta, R)$) be the subgroup generated by all $\tilde{n}_\mu(u)$ (respectively, $\tilde{h}_\mu(u)$), $\mu \in {}^{re}\Delta, u \in R^\times$. Then for $\mu, \nu \in {}^{re}\Delta, a, b \in R, u, v \in R^\times$, the following relations are consequences of (R1′) and (R2′) if rank $\dot{\Delta} \geq 2$, and of (R1′) and (R3′) if rank $\dot{\Delta} = 1$.

(R3′) $\tilde{n}_\mu(u)\tilde{x}_\nu(a)\tilde{n}_\mu(u)^{-1} = \tilde{x}_{r_\mu\nu}(\eta u^{-(\nu|\mu)}a)$

(R4′) $\tilde{n}_\mu(u)\tilde{n}_\nu(v)\tilde{n}_\mu(u)^{-1} = \tilde{n}_{r_\mu\nu}(\eta u^{-(\nu|\mu)}v)$

$\tilde{n}_\mu(u) = \tilde{n}_{-\mu}(u^{-1})$

(R5′) $\tilde{n}_\mu(u)\tilde{h}_\nu(v)\tilde{n}_\mu(u)^{-1} = \tilde{h}_{r_\mu\nu}(\eta u^{-(\nu|\mu)}v)\tilde{h}_{r_\mu\nu}(\eta u^{-(\nu|\mu)})^{-1}$

(R6′) $\tilde{h}_\mu(u)\tilde{x}_\nu(a)\tilde{h}_\mu(u)^{-1} = \tilde{x}_\nu(u^{(\nu|\mu)}a)$

(R7′) $\tilde{h}_\mu(u)\tilde{n}_\nu(v)\tilde{h}_\mu(u)^{-1} = \tilde{n}_\nu(u^{(\nu|\mu)}v)$

$\tilde{h}_\mu(u) = \tilde{h}_{-\mu}(u^{-1})$

(R8′) $\tilde{h}_\mu(u)\tilde{h}_\nu(v)\tilde{h}_\mu(u)^{-1} = \tilde{h}_\nu(u^{(\nu|\mu)}v)\tilde{h}_\nu(u^{(\nu|\mu)})^{-1}$

where if $(\mu \mid \nu) = \pm 1, \eta = \eta(\mu, \nu) = \varepsilon(\mu, \nu)$, and otherwise, $\eta = 1$.

For $u, v \in R^\times$, the element

$$\{u, v\tilde{\}}_\mu := \tilde{h}_\mu(uv)\tilde{h}_\mu(u)^{-1}\tilde{h}_\mu(v)^{-1}$$

is again called a **Steinberg symbol**. Such an element is central in the Steinberg group $St({}^{re}\Delta, R)$. The subgroup generated by all the Steinberg symbols $\{u, v\tilde{\}}_\mu$ is denoted by $\widetilde{C}({}^{re}\Delta, R)$.

We restrict ourselves to Steinberg groups $St(\dot{\Delta}, R_2)$ and $St({}^{re}\Delta, \mathbb{C})$. From the presentations of these two groups, we have

LEMMA 2.2.4. *The map from* $St(^{re}\Delta, \mathbb{C})$ *to* $St(\dot{\Delta}, R_2)$ *defined by*

$$\tilde{x}_\mu(c) \mapsto \hat{x}_\alpha(cs^m t^n)$$

for $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$, *is an isomorphism.*

PROOF: Under this map, $\tilde{n}_\mu(c) \mapsto \hat{n}_\alpha(cs^m t^n)$. Then comparing the relations (R1′), (R2′) and (R3′) respectively with (R1), (R2) and (R3), we get that the map is well defined and the inverse map is given by

$$\hat{x}_\alpha(a) \mapsto \prod_{i,j} \tilde{x}_{\alpha+i\delta_1+j\delta_2}(c_{ij}),$$

where $a = \sum_{i,j} c_{ij} s^i t^j \in R_2$ and the order of the product is arbitrary. Note that elements in the products are commutative to each other by (R2′) (or (R3′) if $\operatorname{rank} \dot{\Delta} = 1$). The well definedness of this inverse map is clear by checking the relations (R1) and (R2) or (R3). □

We shall identify these two groups via this isomorphism. Here is a useful list of correspondences.

LEMMA 2.2.5.

$$\tilde{x}_\mu(a) = \hat{x}_\alpha(as^m t^n)$$
$$\tilde{n}_\mu(c) = \hat{n}_\alpha(cs^m t^n)$$
$$\tilde{h}_\mu(c) = \hat{h}_\alpha(cs^m t^n)\hat{h}_\alpha(s^m t^n)^{-1} = \{s^m t^n, c\}\hat{h}_\alpha(c)$$
$$\tilde{N}(^{re}\Delta, \mathbb{C}) = \hat{N}(\dot{\Delta}, R_2)$$
$$\tilde{H}(^{re}\Delta, \mathbb{C}) \subset \hat{H}(\dot{\Delta}, R_2)$$

for $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$, $a \in \mathbb{C}, c \in \mathbb{C}^\times, m, n \in \mathbb{Z}$. □

Now we conclude this section with a close look at the group $\widehat{C} := \widehat{C}(\dot{\Delta}, R_2)$ for finite root systems $\dot{\Delta}$ with rank at least 2.

Note $R_2^\times = \{cs^m t^n \mid c \in \mathbb{C}^\times, m, n \in \mathbb{Z}\}$. Define

$$\widehat{C}_0 := \{\{c, d\}\hat{} \mid c, d \in \mathbb{C}^\times\}$$
$$\widehat{C}_s := \{\{c, s\}\hat{} \mid c \in \mathbb{C}^\times\}$$
$$\widehat{C}_t := \{\{c, t\}\hat{} \mid c \in \mathbb{C}^\times\}$$
$$\widehat{C}_{s,t} := \{\{s, t\}\hat{}^m \mid m \in \mathbb{Z}\}.$$

Then by Lemma 2.2.2, we have that $\widehat{C}_s$ and $\widehat{C}_t$ are groups isomorphic to $\mathbb{C}^\times$, $\widehat{C}_{s,t}$ is a group isomorphic to $\mathbb{Z}$ with a generator either $\{s, t\}\hat{}$ or $\{t, s\}\hat{}$. It is also clear that $\widehat{C}(\dot{\Delta}, R_2)$ is the direct product of $\widehat{C}_0, \widehat{C}_s, \widehat{C}_t$ and $\widehat{C}_{s,t}$. Hence a typical element of $\widehat{C}(\dot{\Delta}, R_2)$ looks like

$$\hat{z}_0\{c, s\}\hat{}\{d, t\}\hat{}\{s, t\}\hat{}^m, \tag{2.2.6}$$

where $\hat{z}_0 \in \widehat{C}_0$.

Let

$$\widehat{H}_0 := \{\hat{h}_\alpha(c) \mid \alpha \in \dot{\Delta}, c \in \mathbb{C}^\times\}.$$

LEMMA 2.2.6. *Suppose $\dot{\Delta}$ is not of type A_1, i.e. rank $\dot{\Delta} \geq 2$. Then*

(i) $\{c, d\}\tilde{}_\mu = \{c, d\}\hat{}_\alpha$, so $\{c, d\}\tilde{} := \{c, d\}\tilde{}_\mu$ is independent of $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$.

(ii) $\widetilde{H}({}^{re}\Delta, \mathbb{C}) = \widehat{C}_s\widehat{C}_t\widehat{H}_0$.

PROOF: The results follow from the third equation of Lemma 2.2.5 and the centralness of the Steinberg symbols. □

In the sequel, we often simply write $\widehat{N}, \widetilde{N}$ for $\widehat{N}(\dot{\Delta}, R_2)$, $\widetilde{N}({}^{re}\Delta, \mathbb{C})$, etc.

§2.3. Universal Toroidal Groups.

In this section, we follow Chapter 6 of [MP] to define the universal toroidal groups.

First, we recall Corollary 3.4 of [MRY],

LEMMA 2.3.1. *If $\mu \in {}^{re}\Delta$, then $\mathfrak{t}^{\mu} \oplus [\mathfrak{t}^{\mu}, \mathfrak{t}^{-\mu}] \oplus \mathfrak{t}^{-\mu} \simeq \mathfrak{sl}_2(\mathbb{C})$.* □

This little lemma enables us to apply the powerful $\mathfrak{sl}_2$-theory.

Let

$$\mathfrak{h} := \sum_{\mu \in {}^{re}\Delta} [t^{\mu}, t^{-\mu}].$$

Then

$$\mathfrak{h} = 1 \otimes \dot{\mathfrak{h}} \oplus \mathbb{C}\mathfrak{k}_s \oplus \mathbb{C}\mathfrak{k}_t. \tag{2.3.1}$$

DEFINITION 2.3.2. *A $\mathfrak{t}$-module (ρ, M) is said to be* **integrable** *if*

IM1 *for all $\mu \in {}^{re}\Delta$, the elements of $\rho(\mathfrak{t}^{\mu})$ are locally nilpotent.*

IM2 *$\rho(\mathfrak{h})$ is diagonalizable, that is, any element in $\mathfrak{h} = \sum_{\mu \in {}^{re}\Delta} [\mathfrak{t}^{\mu}, \mathfrak{t}^{-\mu}]$ has a set of eigenvectors which consists of a basis of M as a vector space over $\mathbb{C}$.* □

We speak of ρ or M as an **integrable representation** of $\mathfrak{t}$.

The integrable modules form a full subcategory $\mathcal{I}$ of the category of all $\mathfrak{t}$-modules. It is immediate that $\mathcal{I}$ is closed under the formation of direct sums, tensor products, submodules, and quotient modules.

We now construct a group $\overline{G}$ attached to $\mathfrak{t}$ that acts on every integrable $\mathfrak{t}$-module in a way compatible with the action of $\mathfrak{t}$.

Let

$$\widehat{G} = *_{\alpha \in {}^{re}\Delta} t^{\mu}.$$

where t^{μ} is viewed as an abelian additive group. We denote the canonical injective homomorphism $t^{\mu} \to \widehat{G}$ by $\exp^{\wedge}_{\mu}$. We also write the group $\widehat{G}$ multiplicatively so

that for all $\mu \in {}^{re}\Delta$ and $x,\ y \in \mathfrak{t}^{\mu}$

$$\exp_{\mu}^{\wedge}(x+y) = \exp_{\mu}^{\wedge}(x)\exp_{\mu}^{\wedge}(y)$$

and

$$\exp_{\mu}^{\wedge}(-x) = \exp_{\mu}^{\wedge}(x)^{-1}.$$

Since the subgroups $\exp_{\mu}^{\wedge}(\mathfrak{t}^{\mu})$ generate $\widehat{G}$, a typical element of $\widehat{G}$ is a finite product of the form

$$\exp_{\mu_1}^{\wedge}(x_1)\cdots\exp_{\mu_r}^{\wedge}(x_r)$$

for some $r \in \mathbb{N}$ where $\mu_i \in {}^{re}\Delta$ and $x_i \in \mathfrak{t}^{\mu_i}$ for all $1 \leq i \leq r$. If $r = 0$ then this product is understood to be the identity element of $\widehat{G}$.

Let (ρ, M) be an integrable t-module. Then for each $\mu \in {}^{re}\Delta, x \in \mathfrak{t}^{\mu}$, $\exp(\rho(x))$ is a well-defined element of $GL(M)$ by the local nilpotency of $\rho(x)$. The mapping $\exp_{\mu}^{\rho} : \mathfrak{t}^{\mu} \to GL(M)$ given by $x \mapsto \exp\rho(x)$ is a group homomorphism and hence by the universal property of free products, there is a unique homomorphism $\hat{\rho} : \widehat{G} \to GL(M)$ such that

$$\begin{array}{ccc} \mathfrak{t}^{\mu} & \xrightarrow{\exp_{\mu}^{\rho}} & GL(M) \\ \exp_{\mu}^{\wedge} \searrow & & \nearrow \hat{\rho} \\ & \widehat{G} & \end{array}$$

commutes.

Let $\widehat{K} := \cap \ker\hat{\rho}$ where the intersection is taken over all integrable modules (ρ, M) of $\mathfrak{t}$. Define

$$\overline{G} := \widehat{G}/\widehat{K}.$$

We call the group $\overline{G}$ the **universal Chevalley group** associated with the Lie algebra $\mathfrak{t}$. Let $\xi : \widehat{G} \to \overline{G}$ be the natural quotient mapping. For each $x \in \mathfrak{t}^{\mu}$ and $\mu \in \Delta^{re}$, we define $\exp x \in \overline{G}$ by

$$\exp x = \xi(\exp_{\mu}^{\wedge} x).$$

If $\rho : \mathfrak{t} \to \mathfrak{gl}(M)$ is integrable then there exists a unique group representation

$$\boldsymbol{\rho} : \overline{G} \to GL(M)$$

such that the diagram

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\hat{\boldsymbol{\rho}}} & GL(M) \\ \xi \searrow & & \nearrow \boldsymbol{\rho} \\ & \overline{G} & \end{array}$$

commutes.

In this notation, we have for all $\mu \in {}^{re}\Delta$ and for all $x \in \mathfrak{t}^{\mu}$

$$\exp\rho(x) = \boldsymbol{\rho}\exp x. \tag{2.3.2}$$

A particularly important case is that of the adjoint representation

$$ad : \mathfrak{t} \to \mathfrak{gl}(\mathfrak{t}).$$

By the above this leads to a group representation $\mathbf{ad} : \overline{G} \to GL(\mathfrak{t})$. We write Ad instead of $\mathbf{ad}$ in keeping with the standard notation in Lie theory:

$$Ad : \overline{G} \to GL(\mathfrak{t}).$$

Equation (2.3.2) reads

$$\exp(ad\ x) = Ad(\exp x) \quad \text{for all} \quad x \in \mathfrak{t}^{\mu} \quad \text{and} \quad \mu \in {}^{re}\Delta. \tag{2.3.3}$$

Since $\exp(ad\ x) \in \mathrm{Aut}(\mathfrak{t})$ and $\overline{G}$ is generated by $\exp(\mathfrak{t}^{\mu})$, $\mu \in {}^{re}\Delta$, we see that

$$Ad : \overline{G} \to \mathrm{Aut}(\mathfrak{t}).$$

Indeed elements of $Ad(\overline{G})$ are inner autonorphisms of $\mathfrak{t}$.

REMARK 2.3.3. *Let $x \in \overline{G}$ be such that $\rho(x)$ acts as the identity for every integrable representation ρ of $\mathfrak{t}$. Then $x = 1 \in \overline{G}$. Indeed, the definition of $\widehat{K}$ is designed precisely to assume this.*

We show that $\overline{G}$ is a homomorphic image of the Steinberg group $St({}^{re}\Delta, \mathbb{C}) = St(\dot{\Delta}, R_2)$.

Let $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$. Define

$$\bar{x}_\mu(c) = \exp(cs^m t^n \otimes E_\alpha),$$

where E_α's are a part of the Chevalley basis chosen in §2.1. Then $\overline{G}$ is generated by $\bar{x}_\mu(c)$, for all $\mu \in {}^{re}\Delta$ and $c \in \mathbb{C}^\times$.

LEMMA 2.3.4. *With the above choices, $\bar{x}_\mu(c)$'s satisfy the relations (R1') and (R2')(or (R3') if rank $\dot{\Delta} = 1$). Hence the map*

$$\phi : St({}^{re}\Delta, \mathbb{C}) \to \overline{G}$$

defined by $\phi(\tilde{x}_\mu(c)) = \bar{x}_\mu(c)$ is an surjective homomorphism of groups.

PROOF: We only need to show that $\bar{x}_\mu(c)$'s (indeed, $\rho(\bar{x}_\mu(c))$ satisfy (R1') and (R2')(or (R3') if rank $\dot{\Delta} = 1$) for every integrable representation (ρ, M).

Suppose rank $\dot{\Delta} \geq 2$. Let (ρ, M) be any integrable representation of $\mathfrak{t}$. Recall the Lie bracket of $\mathfrak{t}$,

$$[a \otimes x, b \otimes y] = ab \otimes [x, y] + \overline{(da)b}(x \mid y)$$

for all $a, b \in R_2$ and $x, y \in \dot{\mathfrak{g}}$. (R1') is clear from the definition of $\overline{G}$. Let $\mu = \alpha + m\delta_1 + n\delta_2, \nu = \beta + p\delta_1 + q\delta_2 \in {}^{re}\Delta$. If $\mu + \nu \notin {}^{re}\Delta \cup \{0\}$, then $\alpha + \beta \notin \dot{\Delta} \cup \{0\}$. By the orthogonality of the form $(\cdot \mid \cdot)$ on $\dot{\mathfrak{g}}$, we have $(E_\alpha \mid E_\beta) = 0$. Also $[E_\alpha, E_\beta] = 0$. Then $[as^m t^n \otimes E_\alpha, bs^p t^q \otimes E_\beta] = 0$. Then by Campell-Baker-Housdaff formula(see §3 of [St2]) the group commutator

$$(\exp\rho(as^m t^n \otimes E_\alpha), \exp\rho(bs^p t^q \otimes E_\beta))$$

acts on M as identity map. So $\rho(\bar{x}_\mu(a), \bar{x}_\nu(b)) = 1 \in GL(M)$ and $(\bar{x}_\mu(a), \bar{x}_\nu(b)) = 1$ by Remark 2.3.3. Similarly, if $\mu + \nu \in {}^{re}\Delta$,

$$(\exp\rho(as^m t^n \otimes E_\alpha), \exp\rho(bs^p t^q \otimes E_\beta)) = \exp\rho(\varepsilon(\alpha,\beta)ab)$$

since $[E_\alpha, E_\beta] = \varepsilon(\alpha,\beta)E_{\alpha+\beta}$ and $[E_\alpha, E_{\alpha+\beta}] = 0 = [E_\beta, E_{\alpha+\beta}]$. Then (R2′) holds in this case.

If $\operatorname{rank}\dot{\Delta} = 1$, (R3′) will follow from Corollary 2.3.6 below. □

For each $\mu \in {}^{re}\Delta$, let $SL_2^{(\mu)}$ be the subgroup of $\overline{G}$ generated by $\exp\mathfrak{t}^\mu$ and $\exp\mathfrak{t}^{-\mu}$. Note $\{\xi_\mu, \mu, \xi_{-\mu}\}$ is an $\mathfrak{sl}_2$-triplet. We have

PROPOSITION 2.3.5.

(i) Let (ρ, M) be an integrable t*-module and let $\mu \in {}^{re}\Delta$. Then there exists a surjective homomorphism*

$$k_\rho^{(\mu)} : SL_2(\mathbb{C}) \to \rho(SL_2^{(\mu)})$$

such that for all $c \in \mathbb{C}$

$$k_\rho^{(\mu)}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \rho(\exp c\xi_\mu), \quad k_\rho^{(\mu)}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \rho(\exp c\xi_{-\mu}).$$

(ii) There exists a unique isomorphism

$$k^{(\mu)} : SL_2(\mathbb{C}) \to SL_2^{(\mu)}$$

such that for all $c \in \mathbb{C}$

$$k^{(\mu)}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \exp c\xi_\mu, \quad k^{(\mu)}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \exp c\xi_{-\mu}$$

and

$$\rho \circ k^{(\mu)} = k_\rho^{(\mu)}.$$

PROOF: (i) Same as the proof of Proposition 6.1.7(i) of [MP].

(ii) Let $\widehat{SL}_2^{(\mu)}$ be the subgroup of $\widehat{G}$ generated by $\exp_\mu^\wedge g^\mu$ and $\exp_{-\mu}^\wedge g^{-\mu}$. By universal properties $\widehat{SL}_2^{(\mu)} \cong (\exp_\mu^\wedge \mathfrak{t}^\mu) * (\exp_{-\mu}^\wedge \mathfrak{t}^{-\mu})$.

The homomorphisms $\exp_\mu^\wedge c\xi_\mu \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, $\exp_{-\mu}^\wedge c\xi_{-\mu} \mapsto \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ determine a surjective homomorphism $\omega : \widehat{SL}_2^{(\mu)} \to SL_2(\mathbb{C})$. Let $(\rho, M) \in \mathcal{I}$ and consider the mappings $\boldsymbol{k}_\rho^{(\mu)}$ of (i) and $\hat{\boldsymbol{\rho}} : \widehat{G} \to GL(M)$. From $\hat{\boldsymbol{\rho}}(\exp_\mu^\wedge c\xi_\mu) = \exp(\rho c\xi_\mu) = \boldsymbol{k}_\rho^{(\mu)} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \boldsymbol{k}_\rho^{(\mu)} \omega \exp_\mu^\wedge c\xi_\mu$ and a similar calculation for $\exp c\xi_{-\mu}$ we obtain the commutative diagram

$$\begin{array}{ccc} \widehat{SL}_2^{(\mu)} & \xrightarrow{\hat{\boldsymbol{\rho}}} & \hat{\boldsymbol{\rho}}(\widehat{SL}_2^{(\mu)}) \\ & \searrow \omega \qquad \boldsymbol{k}_\rho^{(\mu)} \nearrow & \\ & SL_2(\mathbb{C}) & \end{array}$$

Thus $\ker\hat{\boldsymbol{\rho}} \cap \widehat{SL}_2^{(\mu)} \supset \ker\omega$. This being true for all $(\rho, M) \in \mathcal{I}$ we have $\hat{K} \cap \widehat{SL}_2^{(\mu)} \supset \ker\omega$ and hence an induced homomorphism $\boldsymbol{k}^{(\mu)} : SL_2(\mathbb{C}) \to SL_2^{(\mu)}$ with $\boldsymbol{k}^{(\mu)}\omega = \xi$ on $\widehat{SL}_2^{(\mu)}$ (see above for the definition of $\xi : \widehat{G} \to G$). This leads to the commutative diagram

$$\begin{array}{ccc} \widehat{SL}_2^{(\mu)} & \xrightarrow{\hat{\boldsymbol{\rho}}} & \boldsymbol{\rho}(SL_2^{(\mu)}) = \hat{\boldsymbol{\rho}}(\widehat{SL}_2^{(\mu)}) \\ & \searrow \xi \qquad \boldsymbol{\rho} \nearrow & \\ & SL_2^{(\mu)} & \\ & \omega \searrow \quad \uparrow \boldsymbol{k}^{(\mu)} \quad \nearrow \boldsymbol{k}_\rho^{(\mu)} & \\ & SL_2(\mathbb{C}) & \end{array}$$

Now $\boldsymbol{k}^{(\mu)}$ is surjective since ξ is. Hence it remains to show that $\boldsymbol{k}^{(\mu)}$ is injective. It suffices to find one integrable $\mathfrak{t}$-module such that $\boldsymbol{k}_\rho^{(\mu)}$ is injective.

Choose a positive system $\dot{\Delta}_+$ of $\dot{\Delta}$ such that $\alpha \in \dot{\Delta}_+$, where $\mu = \alpha + m\delta_1 + n\delta_2$. Choose $\lambda \in \dot{P}_+$(the set of dominant weights of $\dot{\Delta}_+$) so that $(\lambda \mid \mu) \in 2\mathbb{Z} + 1$ (this is possible). So $L(\lambda)$, the irreducible highest weight module of $\dot{\mathfrak{g}}$, is an integrable $\dot{\mathfrak{g}}$-moduls. Let v_+ denote a highest weight vector for $L(\lambda)$. Make $L(\lambda)$ into a $\mathfrak{t}$-module

by defining

$$f \otimes x.v = f(1,1)x.v, \quad z.v = 0$$

for all $v \in L(\lambda), x \in \dot{\mathfrak{g}}, f \in R_2$ and $z \in \Omega/dR_2$ where $f(1,1)$ is the evaluation of f at $s = t = 1$. This is well defined and $L(\lambda)$ becomes an integrable t-module. In particular, $L(\lambda)$ is an integrable $\mathfrak{sl}_2^{(\mu)}$-module.

Then

$$M := \sum_{i=0}^{(\lambda|\mu)} \mathbb{C}(\xi_{-\mu})^i.v_+$$

is the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $(\lambda \mid \mu)+1 \in 2\mathbb{Z}$. But the corresponding action of $SL_2(\mathbb{C})$ is faithful (Proposition 2.2.4, Corollary 6 of [MP]). If we denote this t-module by ρ, our conclusion is that $k_\rho^{(\mu)}$ is injective and hence also $k^{(\mu)}$. □

Since

$$k^{(\mu)}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \exp c\xi_\mu = \bar{x}_\mu(c), \quad k^{(\mu)}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \exp c\xi_{-\mu} = \bar{x}_{-\mu}(c)$$

for $c \in \mathbb{C}$, then

$$k^{(\mu)}\begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} = \bar{n}_\mu(c) \quad k^{(\mu)}\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \bar{h}_\mu(c)$$

for $c \in \mathbb{C}^\times$. Hence the following two corollaries are consequences of Proposition 2.3.5(ii) and relative relations in $SL_2(\mathbb{C})$.

COROLLARY 2.3.6. *(R3′) holds for $\bar{x}_\mu(c)$'s for all $\mu \in {}^{re}\Delta$ and $c \in \mathbb{C}^\times$.* □

COROLLARY 2.3.7. *For $\mu \in {}^{re}\Delta$ and $c \in \mathbb{C}^\times$, let $\bar{h}_\mu(c) = \phi(\tilde{h}_\mu(c))$, where ϕ is defined in Lemma 2.3.4. Then*

(R0′) $\quad \bar{h}_\mu(c)\bar{h}_\mu(d) = \bar{h}_\mu(cd),$

for all $c, d \in \mathbb{C}^\times, \mu \in {}^{re}\Delta$. In other words, the Steinberg symbols $\{c, d\}_\mu$ are in the kernel of the map ϕ. □

CONJECTURE 2.3.8. *The universal toroidal group $\overline{G}$ is the abstract group with the generators $\bar{x}_\mu(c)$, $\mu \in {}^{re}\Delta$, $c \in \mathbb{C}$ and relations (R0′), (R1′) and (R2′) (or replacing (R2′) by (R3′) when rank $\dot{\Delta} = 1$).*

By using some results in K-theory and some representations of $\mathfrak{t}$, we will show in §2.6 that the conjecture is true in the cases when $\dot{\Delta}$ is of type $A_\ell, \ell \geq 4$.

§2.4. Vertex Representations of Toroidal Lie Algebras.

In this section, we briefly review the notation and construction of the presentation of $\mathfrak{t}$ and the vertex representations of $\mathfrak{t}$ from [25] and [9]. Then we conclude with Proposition 2.4.4, which will give an explicit formula how the toroidal groups act on the vertex representations and enalbe us to do calculations (cf Proposition 2.5.2). We maintain the notation in §2.1.

Let $A = (A_{ij})_{i,j=0}^{\ell}$ be an indecomposable Cartan matrix of affine type $X_\ell^{(1)}$ ($X = A_\ell, D_\ell, E_\ell$). Let Q_a be the free $\mathbb{Z}$-module on generators $\alpha_0, \ldots, \alpha_\ell$ and identify the affine root system Δ_a defined by A with a subset of Q_a by identifying $\{\alpha_0, \ldots, \alpha_\ell\}$ with a base Π_a of Δ_a. We know [M] that there is a $\mathbb{Z}$-valued symmetric bilinear form $(\cdot \mid \cdot)$ on Q_a for which, after suitable choice of indexing, $(\alpha_i \mid \alpha_j) = A_{ij}$.

Let $\delta := \sum_{i=0}^{\ell} n_i\alpha_i$, $n_i \in \mathbb{Z}_+$, $gcd(n_0, \ldots, n_\ell) = 1$, be the null root. We assume that the notation is chosen so that $n_0 = 1$, so α_0 is an "extension node" and $\dot{A} = (A_{ij})_{i,j=1}^{\ell}$ is of finite type X_ℓ. We assume that $(\cdot \mid \cdot)$ is scaled so that $(\alpha_0 \mid \alpha_0) = 2$. We may view $\dot{Q} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_\ell \subset Q_a$, $\dot{\Pi} := \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta_a$.

We let $\mathfrak{t}(A)$ be the Lie algebra over $\mathbb{C}$ with the following presentation:

$$\textbf{generators}: \mathfrak{k},\quad \alpha_i(k),\quad x_k(\pm\alpha_i)\quad i = 0, \ldots, \ell,\quad k \in \mathbb{Z} \tag{2.4.1}$$

relations:

TA0 $\quad [\mathfrak{k},\ \alpha_i(k)] = [\mathfrak{k},\ x_k(\pm\alpha_i)] = 0$

TA1 $\quad [\alpha_i(k), \alpha_j(m)] = k(\alpha_i \mid \alpha_j)\delta_{k+m,0}\mathfrak{k}$

TA2 $\quad [\alpha_i(k), x_m(\pm\alpha_j)] = \pm(\alpha_i \mid \alpha_j)x_{m+k}(\pm\alpha_j)$

TA3 $\quad [x_m(\alpha_i), x_n(-\alpha_j)] = -\delta_{ij}\left\{\alpha_i(m+n) + \dfrac{2m\delta_{m+n,0}}{(\alpha_i \mid \alpha_i)}\mathfrak{k}\right\}$

TA4 $\quad [x_m(\alpha_i), x_n(\alpha_i)] = 0 = [x_m(-\alpha_i), x_n(-\alpha_i)]$

$$\left.\begin{aligned}(ad\ x_0(\alpha_i))^{-A_{ji}+1}x_m(\alpha_j) = 0\\ (ad\ x_0(-\alpha_i))^{-A_{ji}+1}x_m(-\alpha_j) = 0\end{aligned}\right\} i \neq j$$

for all $i,j = 0,\dots,\ell,\ k,m,n \in \mathbf{Z}$.

Let $\{E_\alpha, \alpha_i \mid \alpha \in \dot{\Delta}, 1 \le i \le \ell\}$ be the Chevalley basis of $\dot{\mathfrak{g}}$ associated to the cocycle $\dot{\varepsilon}$ as chosen in §2.1. Set

$$e_i := E_{\alpha_i}, f_i := -E_{-\alpha_i}, h_i = \alpha_i,\ 1 \le i \le \ell.$$

Then we easily have commutation relations:

$$\begin{aligned}
&\text{(i)} \quad [h_i, e_j] = A_{ji} \\
&\text{(ii)} \quad [h_i, f_j] = -A_{ji} \\
&\text{(iii)} \quad [e_i, f_j] = \delta_{ji} h_i \\
&\text{(iv)} \quad (ad\, e_i)^{-A_{ji}+1} e_j = 0 \qquad i \ne j; \\
&\text{(v)} \quad (ad\, f_i)^{-A_{ji}+1} f_j = 0, \qquad i \ne j.
\end{aligned}$$

for all $1 \le i,j \le \ell$. It is well known that relations (i)-(v) are the defining relations of $\dot{\mathfrak{g}}$. Let θ be the highest root of $\dot{\Delta}$ relative to $\dot{\Pi} = \{\alpha_1,\dots,\alpha_\ell\}$ and let $e_0 := E_{-\theta} \in \dot{\mathfrak{g}}^{-\theta}$, $f_0 := -E_\theta \in \dot{\mathfrak{g}}^{\theta}$. Then $\{e_0, h_0, f_0\}$ is an $\mathfrak{sl}_2$-triplet where $h_0 := -\sum_{i=1}^{\ell} n_i h_i$.

PROPOSITION 2.4.1. *The mapping*

$$\begin{aligned}
t &\mapsto \overline{s^{-1}ds} \\
x_m(\alpha_i) &\mapsto s^m \otimes e_i \quad i = 1,\dots,\ell \\
x_m(-\alpha_i) &\mapsto -s^m \otimes f_i \quad i = 1,\dots,\ell \\
x_m(\alpha_0) &\mapsto s^m t \otimes e_0 \\
x_m(-\alpha_0) &\mapsto -s^m t^{-1} \otimes f_0 \\
\alpha_i(k) &\mapsto s^k \otimes h_i \qquad i = 1,\dots,\ell \\
\alpha_0(k) &\mapsto s^k \otimes h_0 + \overline{s^k t^{-1} dt},
\end{aligned}$$

defines an isomorphism from $\mathfrak{t}(A) \to \mathfrak{t}$. □

For more details, see §3 of [MRY]. From the above discusion, we see that the variable t plays the role of first affinization to get the affine Lie algebra from $\dot{\mathfrak{g}}$. So we may identify δ to δ_2, and then $Q_a = \dot{Q} + \mathbf{Z}\delta_2$ can be viewed as a sublattice of Q.

Now we construct vertex representations for $\mathfrak{t}$. We assume that the lattice Q is a sublattice of a lattice Γ admitting a non-singular $\mathbf{Z}$-valued symmetric bilinear form $(\cdot \mid \cdot)$ that restricts to the previously chosen form on Q. Such a lattice Γ can be constructed in many ways. For example, $\Gamma^{[2]}$ of §3.1 works. Let $\mathfrak{k} := \mathbf{C} \otimes_{\mathbf{Z}} \Gamma$ and for each $k \in \mathbf{Z}$ take an isomorphic copy $\mathfrak{k}(k)$ of $\mathfrak{k}$. Denote the isomorphism by $\alpha \mapsto \alpha(k)$. Define a Heisenberg algebra $\hat{\mathfrak{b}} := \bigoplus_{k \in \mathbf{Z}} \mathfrak{k}(k) \oplus \mathbf{C}\mathfrak{c}$ with Lie bracket

$$[\mathfrak{c}, \hat{\mathfrak{b}}] := 0$$

$$[\alpha(k), \beta(m)] := k(\alpha \mid \beta)\delta_{k+m,0}\mathfrak{c}$$

We let σ be the conjugate linear involution on $\hat{\mathfrak{b}}$ defined by $\alpha(n) \mapsto \alpha(-n)$ for all $\alpha \in \mathfrak{k}_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Z}} \Gamma$. Define

$$\mathfrak{b} := \sum_{k \in \mathbf{Z} \setminus \{0\}} \mathfrak{k}(k) + \mathbf{C}\mathfrak{c} \subset \hat{\mathfrak{b}}$$

$$\mathfrak{b}_{\pm} := \sum_{k \gtrless 0} \mathfrak{k}(k)$$

The Fock space representation of $\mathfrak{b}$ is the symmetric algebra $S(\mathfrak{b}_-)$ of $\mathfrak{b}_-$ together with the action of $\mathfrak{b}$ on $S(\mathfrak{b}_-)$ defined by

$\mathfrak{c}$ acts as 1

$a(-m)$ acts as multiplication by $a(-m)$, $m > 0$

$a(m)$ acts as the unique derivation on $S(\mathfrak{b}_-)$ for which $b(-n) \mapsto \delta_{m-n,0} m(a \mid b)$

for all $a, b \in \mathfrak{k}$, $m, n > 0$.

$S(\mathfrak{b}_-)$ affords an irreducible representation of $\mathfrak{b}$. Let $\varepsilon : Q \times Q \to \{\pm 1\}$ be a bimultiplicative map satisfying **CC(i), CC(ii), CC(iii)** (see §2.1). Condition (iii)

makes it easy to determine such an ε by using a corresponding cocycle on $\dot{Q}$. For our purposes, we will use the one chosen in §2.1.

We will asume that *ε can be extended to a bimultiplicative map*

$$\varepsilon : Q \times \Gamma \to \{\pm 1\}.$$

No further assumptions on ε are needed.

For each $\gamma \in \Gamma$ let e^γ be a symbol and form the vector space $\mathbb{C}[\Gamma]$ with basis $\{e^\gamma\}$ over $\mathbb{C}$. Following Borcherds [Bo] we define a twisted group algebra structure on $\mathbb{C}[Q]$ by

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}.$$

We define

$$V_\Gamma := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} S(\mathfrak{b}_-).$$

V_Γ is made into an $\hat{\mathfrak{b}}$-module by defining

$$a(m).(e^\gamma \otimes u) := e^\gamma \otimes a(m) \cdot u \quad \text{if } m \neq 0$$

$$a(0).(e^\gamma \otimes u) := (a \mid \gamma) e^\gamma \otimes u \quad \text{if } m = 0.$$

Let z be a complex-valued variable and let $\alpha \in Q$. Define

$$T_\pm(\alpha, z) := -\sum_{n \gtrless 0} \frac{1}{n} \alpha(n) z^{-n}.$$

Then the vertex operator for α on V_Γ is defined as

$$X(\alpha, z) := z^{\frac{(\alpha|\alpha)}{2}} \exp T(\alpha, z)$$

where

$$\exp T(\alpha, z) := \exp T_-(\alpha, z) e^\alpha z^{\alpha(0)} \exp T_+(\alpha, z)$$

and the operator $z^{\alpha(0)}$ is defined by

$$z^{\alpha(0)} e^\gamma \otimes u := z^{(\alpha|\gamma)} e^\gamma \otimes u.$$

Strictly, for each $z \in \mathbb{C}^\times$ the operator $X(\alpha, z)$ maps V_Γ into the space $C[\Gamma] \otimes \widehat{S(\mathfrak{b}_-)}$ where $\widehat{S(\mathfrak{b}_-)} = \coprod_n S(\mathfrak{b}_-)^n$ is the completion to formal power series of $S(\mathfrak{b}_-)$. However $X(\alpha, z)$ can be formally expanded in powers of z to give

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

and the "moments" $X_n(\alpha)$ are indeed operators on V_Γ. Moreover for all $v \in S(\mathfrak{b}_-)$, $X_n(\alpha)(e^\gamma \otimes v) = e^{\gamma+\alpha} \otimes v'$ where $v' \in S(\mathfrak{b}_-)$ is obtained from v by applying some polynomial expression in the operators $a(m)$, $a \in Q$, $m \in \mathbb{Z}\backslash\{0\}$.

The powers of z in $X(\alpha, z)$ are so construed that $X_n(\alpha)$ is an operator of degree $(-n, \alpha)$ relative to the $\mathbb{Z} \times \Gamma$ grading on V_Γ for which

$$\deg\left(e^\gamma \otimes a_k(-n_k) \dots a_1(-n_1)\right) = \left(\frac{1}{2}(\gamma \mid \gamma) + n_1 + \cdots + n_k, \gamma\right).$$

Assigning $X_n(\alpha)$ degree (n, α) would seem more natural for what is to come but the grading on V_Γ is well established in the literature.

V_Γ is also graded by Γ:

$$V_\Gamma = \bigoplus_{\lambda \in \Gamma} V^\lambda \tag{2.4.2}$$

where $V^\lambda := e^\lambda \otimes S(\mathfrak{b}_-)$ for all $\alpha \in \Gamma$. And then

$$X_m(\alpha) V^\lambda \subset V^{\lambda+\alpha}. \tag{2.4.3}$$

The basic commutation relations for the operators $\alpha(k)$ and $X_m(\alpha)$ on V_Γ are these ([FK], [GO], [MP]):

CR0 $\quad [\alpha(k), X_n(\beta)] = (\alpha \mid \beta) X_{m+k}(\beta)$

CR1 $\quad [X_m(\alpha), X_n(\beta)] = 0$ if $(\alpha \mid \beta) \geq 0$

CR2 $\quad [X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha, \beta) X_{m+n}(\alpha + \beta)$ if $(\alpha \mid \beta) = -1$

CR3 $\quad [X_m(\alpha), X_n(-\alpha)] = -\alpha(m+n) - m\delta_{m+n,0}\not{c}$ if $(\alpha \mid \alpha) = 2$.

In fact **CR3** is a special case of

CR4 $$[X_m(\alpha), X_n(\beta)] = \varepsilon(\alpha,\beta)\{mX_{m+n}(\alpha+\beta) + \sum_{k\in\mathbb{Z}} :\alpha(k)X_{m+n-k}(\alpha+\beta):\}.$$

where the normal ordering symbols : : indicate that the operators are to be applied in the order of increasing degree. In particular here

$$:\alpha(k)X_{m+n-k}(\beta): = \begin{cases} \alpha(k)X_{m+n-k}(\beta) & \text{if } k \le m+n-k \\ X_{m+n-k}(\beta)\alpha(k) & \text{if } k > m+n-k. \end{cases}$$

This result is itself a special case of the general commutator formula (8.4.43) of [FLM]. It is also easily derivable from the technique of residue calculus.

The next proposition is a restatement of Proposition 4.3 of [MRY].

PROPOSITION 2.4.2. *The Lie algebra* $\mathfrak{t}_\Gamma$ *of operators on* V_Γ *generated by the operators* $X_m(\alpha + n\delta_2)$, $m, n \in \mathbb{Z}$, $\alpha \in \dot{\Delta}$, *is isomorphic to* $\mathfrak{t}(A)$ *via the uniquely defined map* $\rho : (\mathfrak{t} \simeq)\mathfrak{t}(A) \to \mathfrak{t}_\Gamma$ *for which* $x_m(\pm\alpha_i) \mapsto X_m(\pm\alpha_i)$, $\mathfrak{k} \mapsto 1; 0 \le i \le \ell$. *In other words, together with the isomorphism in Proposition 2.4.1,* (ρ, V_Γ) *is a faithful representation of* $\mathfrak{t}$. □

PROPOSITION 2.4.3. *The representation* (ρ, V_Γ) *of* $\mathfrak{t}$ *is integrable.*

PROOF: See [EMY]. □

We identify $\mathfrak{t}$ with $\mathfrak{t}_\Gamma$ by using the isomorphism ρ, and then the operator $X_m(\alpha + n\delta_2)$ is thought of as an element of $\mathfrak{t}^{\alpha+m\delta_1+n\delta_2}$. So we have

$$s^m \otimes E_{\alpha_i} = x_m(\alpha_i) = X_m(\alpha_i) \tag{2.4.4}$$

$$s^m \otimes E_{-\alpha_i} = x_m(-\alpha_i) = X_m(-\alpha_i) \tag{2.4.5}$$

$$s^m t \otimes E_{-\theta} = x_m(-\theta + \delta_2) = X_m(-\theta + \delta_2) \tag{2.4.6}$$

$$s^m t \otimes E_{\theta} = x_m(\theta - \delta_2) = X_m(\theta - \delta_2) \tag{2.4.7}$$

for all $m \in \mathbb{Z}, 1 \le i \le \ell$ and

PROPOSITION 2.4.4. $X_m(\alpha + n\delta_2) = s^m t^n \otimes E_\alpha$, *for all* $\alpha \in \dot{\Delta}, m, n \in \mathbb{Z}$.

PROOF: We prove this in a few steps.

Step 1. Show $X_m(\alpha) = s^m \otimes E_\alpha$ for all $\alpha \in \dot{\Delta}, m \in \mathbb{Z}$.

Suppose α is a positive root. We use induction on the height of α, $ht(\alpha)$. If $ht(\alpha) = 1$, this is (2.4.4). If $ht(\alpha) > 1$, there is a simple root $\beta \in \dot{\Pi}$ such that $\alpha - \beta \in \dot{\Delta}$, that is $(\alpha \mid \beta) = 1$. Then by (CR2) and induction ($ht(\alpha - \beta) < ht(\alpha)$),

$$\begin{aligned} X_m(\alpha) &= X_m(\alpha - \beta + \beta) \\ &= \varepsilon(\alpha - \beta, \beta)[X_m(\alpha - \beta), X_0(\beta)] \\ &= \varepsilon(\alpha - \beta, \beta)[s^m \otimes E_{\alpha-\beta}, 1 \otimes E_\beta] \\ &= s^m \otimes E_\alpha. \end{aligned}$$

If α is negative, (2.4.5) and similar proof give the result.

Step 2. Show that $X_m(\alpha \pm \delta_2) = s^m t^{\pm 1} \otimes E_\alpha$(respectively) for all $\alpha \in \dot{\Delta}, m \in \mathbb{Z}$.

First of all we have that if $\beta, \gamma \in \dot{\Delta}$ and $(\beta \mid \gamma) = -1$, then

$$X_m(\beta \pm \delta_2) = s^m t^{\pm 1} \otimes E_\beta$$

implies

$$X_m(\gamma \pm \delta_2) = s^m t^{\pm 1} \otimes E_\gamma,$$

respectively. Indeed, by Step 1, (CR1) and (CR2)

$$\begin{aligned} s^m t^{\pm 1} \otimes E_\gamma &= [1 \otimes E_\gamma, s^m t^{\pm 1} \otimes \beta] = [1 \otimes E_\gamma, [s^m t^{\pm 1} \otimes E_{-\beta}, 1 \otimes E_\beta]] \\ &= [X_0(\gamma), [X_m(-\beta \pm \delta_2), X_0(\beta)]] = [X_0(\gamma), [X_m(-\beta \pm \delta_2), X_0(\beta)]] \\ &= [X_m(-\beta \pm \delta_2), [X_0(\gamma), X_0(\beta)]] = [X_m(-\beta \pm \delta_2), \varepsilon(\gamma, \beta) X_0(\beta + \gamma)] \\ &= \varepsilon(-\beta \pm \delta_2, \beta + \gamma)\varepsilon(\gamma, \beta) X_m(\gamma \pm \delta_2) = X_m(\gamma \pm \delta_2). \end{aligned}$$

Now If rank $\dot{\Delta} \geq 2$, then for any two roots β and γ of $\dot{\Delta}$ there is a chain of roots $\beta_0 = \beta, \beta_1, \cdots, \beta_k = \gamma$ of $\dot{\Delta}$ with $(\beta_i \mid \beta_{i+1}) = -1, 0 \leq i \leq k-1$. Thus using (2.4.6) and (2.4.7), we get Step 2.

By (CR4), we have

$$[[X_k(\alpha + n\delta_2), X_m(-\alpha + \delta_2)], X_0(-\alpha)] = [[X_k(\alpha + (n+1)\delta_2), X_m(-\alpha)], X_0(-\alpha)] \tag{2.4.8}$$

since $X_m(n\delta_2)$ is central in t. Taking $n = 0$, we see from (2.4.8) that $X_m(\alpha + \delta_2) = s^m t^{-1} \otimes E_\alpha$ implies $X_m(-\alpha + \delta_2) = s^m t^{-1} \otimes E_{-\alpha}$. Similarly $X_m(\alpha - \delta_2) = s^m t \otimes E_\alpha$ implies $X_m(-\alpha - \delta_2) = s^m t \otimes E_{-\alpha}$. This covers the rank $\dot{\Delta} = 1$ case and proves Step 2.

Step 3. Show that $X_m(\alpha + n\delta_2) = s^m t^n \otimes E_\alpha$ (respectively) for all $\alpha \in \dot{\Delta}, m, n \in \mathbb{Z}$.

Let $X_m(\alpha + n\delta_2) = c_{m,n,\alpha} s^m t^n \otimes E_\alpha$ for some $c_{m,n,\alpha} \in \mathbb{C}^\times$. Without loss of generality, we assume $n > 0$. If we assume $c_{k,n,\alpha} = 1$, then by previous steps and (2.4.8), we have $c_{k,n+1,\alpha} = 1$. Then by induction , we get Step 3. Hence this completes the proof of the proposition. □

This proposition will be used in Proposition 2.5.2.

§2.5. Toroidal Groups Arising from Vertex Representations.

In the last section, for a certain lattice Γ we constructed the vertex representation V_Γ. Since the representation V_Γ for $\mathfrak{t}$ is integrable, by exponentiation we see for each $\mu \in {}^re\Delta$ that $\exp \mathfrak{t}^\mu$ is a subgroup of $GL(V_\Gamma)$. Then we call G_Γ, the group generated by $\exp \mathfrak{t}^\mu$ for all $\mu \in {}^re\Delta$, a **toroidal group** *associated with the vertex representation* V_Γ.

Let

$$x_\mu(c) := x_\alpha(cs^m t^n) := \exp(cs^m t^n \otimes E_\alpha) \in GL(V_\Gamma)$$

for all $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^re\Delta, \alpha \in \dot{\Delta}, m, n \in \mathbb{Z}$. By the definition of the universal toroidal groups, we have that $\bar{x}_\mu(c) \mapsto x_\mu(c)$ defines a surjective homomorphism from $\overline{G}$ onto G_Γ, denoted by ψ. Let $\Phi = \psi\phi$ be the composition map

$$\Phi : St({}^re\Delta, \mathbb{C}) = St(\dot{\Delta}, R_2) \xrightarrow{\phi} \overline{G} \xrightarrow{\psi} G_\Gamma.$$

We use the same notation without *overhats* for objects (elements and subsets) associated with $St(\dot{\Delta}, R_2)(= St({}^re\Delta, \mathbb{C}))$ to denote the images under Φ, respectively. Thus, for example, $n_\alpha(u)$ means $\Phi(\hat{n}_\alpha(u))$.

From now on, we assume the lattice Γ satisfies two additional conditions as follows:

(Γ1) Let $c_1, \cdots, c_\ell \in \mathbb{C}^\times$. If

$$\prod_{i=1}^{\ell} c_i^{(\alpha_i|\lambda)} = 1$$

for all $\lambda \in \Gamma$, then $c_i = 1$.

(Γ2) There is an element $\lambda_0 \in \Gamma$ such that $(\lambda_0 \mid \delta_2) = 1$ and $(\lambda_0 \mid \alpha) = 0$ for all $\alpha \in \dot{\Delta}$.

The purpose in this is to enlarge the vertex representation V_Γ so that some elements in G_Γ can be distinguished easily. This will become clear soon in the proof of Lemma 2.5.7. $\Gamma^{[2]}$ of §3.1 is an example of such a lattice.

We will calculate the actions of elements of G_Γ on V_Γ.

For each $\alpha \in \dot{\Delta}, m, n \in \mathbf{Z}$, define a map $r^{(m)}_{\alpha+n\delta_2}$ on Γ by

$$r^{(m)}_{\alpha+n\delta_2}\lambda := r_{\alpha+n\delta_2}\lambda - m(\alpha + n\delta_2), \tag{2.5.1}$$

where $r_{\alpha+n\delta_2}$ is the reflection in $\alpha + n\delta_2$ relative to the form $(\cdot \mid \cdot)$ on Γ defined by $r_{\alpha+n\delta_2}\lambda := \lambda - (\lambda \mid \alpha + n\delta_2)(\alpha + n\delta_2)$. We denote by t_q the translation map by q on Γ . Then

$$r^{(m)}_{\alpha+n\delta_2} = t_{-m(\alpha+n\delta_2)} \cdot r_{\alpha+n\delta_2} = r_{\alpha+n\delta_2} \cdot t_{m(\alpha+n\delta_2)} \tag{2.5.2}$$

Here is a formula to be used in the proof of Proposition 2.5.3. For any $\beta \in Q_a$ with $(\beta \mid \beta) = 2$,

$$r_{\beta+i\delta_2} r_\beta r_{\beta+j\delta_2} = r_{\beta+(i+j)\delta_2}. \tag{2.5.3}$$

In fact, for any $\lambda \in \Gamma$

$$r_\beta r_{\beta+j\delta_2}\lambda = \lambda + (\lambda \mid j\delta_2)\beta - (\lambda \mid \beta + j\delta_2)j\delta_2$$

and

$$\begin{aligned} r_{\beta+i\delta_2} r_{\beta+(i+j)\delta_2}\lambda &= \lambda + (\lambda \mid j\delta_2)(\beta + i\delta_2) - (\lambda \mid \beta + i\delta_2 + j\delta_2)j\delta_2 \\ &= \lambda + (\lambda \mid j\delta_2)\beta - (\lambda \mid \beta + j\delta_2)j\delta_2. \end{aligned}$$

So $r_\beta r_{\beta+j\delta_2} = r_{\beta+i\delta_2} r_{\beta+(i+j)\delta_2}$ and (2.5.3) holds.

Let W be the subgroup of the group of all bijective maps on Γ generated by all $r^{(m)}_{\alpha+n\delta_2}$, for all $\alpha \in \dot{\Delta}, m, n \in \mathbf{Z}$. For later use, we also write $r^{(m,n)}_\alpha$ for $r^{(m)}_{\alpha+n\delta_2}$.

LEMMA 2.5.1. *$W \simeq W_a \ltimes Q_a$, where W_a is the affine Weyl group and Q_a acts on Γ as translations $t_q, q \in Q_a$.*

PROOF: Since

$$w.t_q.w^{-1} = t_{wq} \quad w \in W_a, q \in Q_a,$$

then W_a stabilizes Q_a under conjugation and Q_a is a normal subgroup of W. The equation (2.5.2) implies W is generated by W_a and Q_a. It remains to show $W_a \cap Q_a = 1$. Suppose $w = t_q \in W_a \cap Q_a$. Then for all $\lambda \in \Gamma$, $w\lambda = \lambda + q$. Take $\lambda = \delta_2$. Since W_a stabilizes δ_2, then $q = 0$. Then $w = t_0 = 1$. The result follows. □

We call W a **toroidal Weyl group**. From Proposition 3.1.12, we see that $W \simeq W^{[2]}$ there.

Since $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$, we can extend linearly the map $r_{\alpha+n\delta_2}$ on Γ to a $\mathbb{C}$-linear map on $\mathfrak{t}$, hence on $\mathfrak{b}$, and an (algebra) automorphism on $S(\mathfrak{b}_-)$.

The following proposition from [EMY] plays a major role on our calculations.

PROPOSITION 2.5.2. *Let $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta, \alpha \in \dot{\Delta}, u = cs^m t^n \in R_2^\times$. Then for $\lambda \in \Gamma, y \in S(\mathfrak{b}_-)$,*

$$n_\alpha(u).e^\lambda \otimes y = [c\varepsilon(\alpha + n\delta_2, \lambda)]^{-(\alpha+n\delta_2|\lambda)-m} e^{r^{(m)}_{\alpha+n\delta_2}\lambda} \otimes r_{\alpha+n\delta_2} y,$$

where $y \in S(\mathfrak{b}_-)$.

PROOF: After the identification between vertex operators and elements in $\mathfrak{t}$ in Proposition 2.4.4, this is Proposition 4.12 in [EMY]. □

Then we have

PROPOSITION 2.5.3. *Let $\alpha \in \dot{\Delta}, m, n, p, q \in \mathbb{Z}, u = cs^m t^n, v = ds^p t^q \in R_2^\times, y \in S(\mathfrak{b}_-), \lambda \in \Gamma$. Then*

(i) $h_\alpha(u).e^\lambda \otimes y$

$$= [-\varepsilon(\alpha, \lambda)]^{-(\alpha|\lambda)} [c\varepsilon(\alpha + n\delta_2, r_\alpha\lambda)]^{-(\alpha+n\delta_2|r_\alpha\lambda)-m} e^{r^{(m)}_{\alpha+n\delta_2} r_\alpha \lambda} \otimes r_{\alpha+n\delta_2} r_\alpha y,$$

$$h_\alpha(u)^{-1}.e^\lambda \otimes y = n_\alpha(1)n_\alpha(-u).e^\lambda \otimes y$$

$$= [-c\varepsilon(\alpha + n\delta_2, \lambda)]^{-(\alpha+n\delta_2|\lambda)-m} \varepsilon(\alpha, r^{(m)}_{\alpha+n\delta_2}\lambda)^{-(\alpha|\lambda)} e^{r_\alpha r^{(m)}_{\alpha+n\delta_2}\lambda} \otimes r_\alpha r_{\alpha+n\delta_2} y,$$

$$h_\alpha(c).e^\lambda \otimes y = c^{(\alpha|\lambda)} e^\lambda \otimes y$$

(ii) $\{u,v\}_\alpha . e^\lambda \otimes y$

$$= (-1)^{[n(\delta_2|\lambda)+m][q(\delta_2|\lambda)+p]}\varepsilon(\delta_2,\lambda)^{mq-pn}c^{-q(\delta_2|\lambda)-p}d^{n(\delta_2|\lambda)+m}\ e^{\lambda+(mq-pn)\delta_2}\otimes y.$$

So $\{u,v\} := \{u,v\}_\alpha$ *is independent of* α.

(iii) $\{c,d\} = 1 \in G_\Gamma$

$$\{c,s\}e^\lambda \otimes y = c^{-1}e^\lambda \otimes y.$$

$$\{c,t\}e^\lambda \otimes y = c^{-(\delta_2|\lambda)}e^\lambda \otimes y.$$

$$\{s,t\}e^\lambda \otimes y = (-1)^{(\delta_2|\lambda)}\varepsilon(\delta_2,\lambda)e^{\lambda+\delta_2}\otimes y.$$

PROOF: (i) and (iii) are immediate from Proposition 2.5.2 and (ii), respectively. The proof of (ii) is organized in the following three steps. We proceed the proof as follows:

Step 1: $\{u,v\}_\alpha \mathbb{C}e^\lambda \otimes y = \mathbb{C}e^{\lambda+(mq-pn)\delta_2}\otimes y$.

Step 2: $\{u,v\}_\alpha . e^\lambda \otimes y = Kc^{-q(\delta_2|\lambda)-p}d^{n(\delta_2|\lambda)+m}e^{\lambda+(mq-pn)\delta_2}\otimes y$, for some $K \in \{\pm 1\}$.

Step 3: the sign K in step 2 is equal to

$$(-1)^{[n(\delta_2|\lambda)+m][q(\delta_2|\lambda)+p]}\varepsilon(\delta_2,\lambda)^{mq-pn}.$$

Since $\{u,v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}$, then by (i) we have

$$\{u,v\}_\alpha \mathbb{C}e^\lambda \otimes y = \mathbb{C}e^{\lambda'}\otimes y'$$

where

$$\begin{aligned}\lambda' &= r^{(m+p)}_{\alpha+(n+q)\delta_2} r_\alpha r_\alpha r^{(m)}_{\alpha+n\delta_2} r_\alpha r^{(p)}_{\alpha+q\delta_2}\lambda \\ &= r^{(m+p)}_{\alpha+(n+q)\delta_2} r^{(m)}_{\alpha+n\delta_2} r_\alpha r^{(p)}_{\alpha+q\delta_2}\lambda\end{aligned} \tag{2.5.4}$$

$$y' = r_{\alpha+(n+q)\delta_2} r_\alpha r_\alpha r_{\alpha+n\delta_2} r_\alpha r_{\alpha+q\delta_2} y \tag{2.5.5}$$

$$= r_{\alpha+(n+q)\delta_2} r_{\alpha+n\delta_2} r_\alpha r_{\alpha+q\delta_2} y.$$

So $y' = y$ by applying (2.5.3) to (2.5.5). Moreover, continued from (2.5.4)

$$\begin{aligned}\lambda' &= r_{\alpha+(n+q)\delta_2}\{r_{\alpha+n\delta_2} r_\alpha r_{\alpha+q\delta_2}(\lambda + p(\alpha+q\delta_2)) \\ &\quad - m(\alpha+n\delta_2) + (m+p)(\alpha+(n+q)\delta_2)\} \\ &= \lambda + p(\alpha+q\delta_2) + r_{\alpha+(n+q)\delta_2}(mq\delta_2 + p(\alpha+(n+q)\delta_2)) \\ &= \lambda + (mq-pn)\delta_2\end{aligned}$$

This proves Step 1.

Using formulas in (i) (writing $\pm$ for the sign parts) and Step 1, we have

$$\{u,v\}_\alpha . e^\lambda \otimes y \tag{2.5.6}$$

$$= \pm (cd)^{-(\alpha+(n+q)\delta_2 | r_\alpha \lambda_2) - m - p} c^{-(\alpha+n\delta_2|\lambda_1)-m} d^{-(\alpha+q\delta_2|\lambda)-p}\ e^{\lambda+(mq-pn)\delta_2} \otimes y,$$

where

$$\lambda_1 = r_\alpha r^{(p)}_{\alpha+q\delta_2},$$

$$\lambda_2 = r_\alpha r^{(m)}_{\alpha+n\delta_2} \lambda_1 = r_\alpha r^{(m)}_{\alpha+n\delta_2} r_\alpha r^{(p)}_{\alpha+q\delta_2} \lambda.$$

Then in (2.5.6),

the power of c

$$\begin{aligned}&= -(\alpha+(n+q) \mid r^{(m)}_{\alpha+n\delta_2}\lambda_1) - m - p - (\alpha+n\delta_2 \mid \lambda_1) - m \\ &= (alpha + n\delta_2 - q\delta_2 \mid \lambda_1) + (\alpha \mid m\alpha) - 2m - p - (\alpha+n\delta_2 \mid \lambda_1) \\ &= -(q\delta_2 \mid \lambda_1) - p = -q(\delta_2 \mid \lambda) - p\end{aligned}$$

Similarly, the power of d in (2.5.6) is equal to $n(\delta_2 \mid \lambda) + m$. Referring back to (2.5.6), we get that Step 2 holds.

We have to count *signs* in details to prove Step 3. The counting uses the cocycle condition CC(i)-(iii) all the time. We will omit this straightforward but lengthy details. □

PROPOSITION 2.5.4. *(i) The restriction of the map $\Phi : St(\dot{\Delta}, R_2) \to G_\Gamma$ on $\hat{U}^+$ is an isomorphism onto U^+.*

(ii) $\widehat{H} \lhd \widehat{N}$, $\overline{H} \lhd \overline{N}$ and $H \lhd N$.

(iii) $\widehat{N}/\widehat{H} \simeq \overline{N}/\overline{H} \simeq N/H \simeq \dot{W}$, where $\dot{W}$ is the finite Weyl group of $\dot{\Delta}$.

PROOF: (ii) follows from (R5) and (R5′) directly. Replacing the representation by V_Γ in the proofs of Lemma 17 and Lemma 22 of [St2] and using (2.4.3), we may conclude (i) and (iii). □

What is $\widetilde{N}/\widetilde{H}$? It turns out that it is isomorphic to the toroidal Weyl group W. We show this in Proposition 2.5.10

LEMMA 2.5.5. *Let $\alpha \in \dot{\Delta}$ and define $\underline{\hat{h}}_\alpha$ to be the subgroup of $St(\dot{\Delta}, R_2)$ generated by all $\hat{h}_\alpha(u), u \in R_2^\times$. Then*

(i) Each $\underline{\hat{h}}_\alpha$ is normal in $\widehat{H}$

(ii) $\widehat{H} = \prod_{i=1}^{\ell} \underline{\hat{h}}_{\alpha_i}$, where $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ is the set of simple roots.

PROOF: See Lemma 38 of [St2]. □

LEMMA 2.5.6. *Let $\widehat{C}$ be the group $\widehat{C}(\dot{\Delta}, R_2)$ of Steinberg symbols. Then, for each $\alpha \in \dot{\Delta}$,*

$$\underline{\hat{h}}_\alpha \subset \{\hat{h}_\alpha(u) \mid u \in R^\times\} \cdot \widehat{C}.$$

PROOF: Note that $\hat{h}_\alpha(u)\hat{h}_\alpha(v) = \hat{h}_\alpha(uv)\{u,v\}_\alpha^{\widehat{}\,-1}$, and $\{u,v\}_\alpha$ is central. Since a typical element of $\underline{\hat{h}}_\alpha$ is of form $\underline{\hat{h}}_\alpha(u_1)\cdots\underline{\hat{h}}_\alpha(u_k)$, then it can be formatted to $\underline{\hat{h}}_\alpha(u_1\cdots u_k)\cdot\hat{z}$ for some $\hat{z} \in \widehat{C}$. □

Let $\dot{\Pi} = \{\alpha_1, \cdots, \alpha_\ell\}$ be the set of simple roots of $\dot{\Delta}$. For the simplicity of notation, we write $r_i^{(m,n)}$ for $r_{\alpha_i+n\delta_2}^{(m)}$.

LEMMA 2.5.7. *Let $\mu = k_1\alpha_1 + \cdots + k_\ell\alpha_\ell + k_0\delta_2 \in Q$. Suppose $w \in \dot{W}, m_i, n_i \in$*

$\mathbf{Z}, 1 \le i \le \ell$ *such that*

$$w\lambda \equiv r_1^{(m_1,n_1)} r_1 r_2^{(m_2,n_2)} r_2 \cdots r_\ell^{(m_\ell,n_\ell)} r_\ell(\lambda+\mu) \qquad \text{mod } \mathbf{Z}\delta_2 \tag{2.5.7}$$

for all $\lambda \in \Gamma$. *Then all* $m_i = k_i, n_i = 0$ *and* $w = 1$.

PROOF: Let $\lambda \in \Gamma$. We have

$$r_i^{(m_i,n_i)} r_i \lambda = \lambda + \nu_i(\lambda)$$

where

$$\nu_i(\lambda) = -[(\lambda \mid n_i\delta_2) + m_i]\alpha_i - [-(\lambda \mid \alpha_i + n_i\delta_2) + m_i]n_i\delta_2 \in \mathbf{Z}\delta_2 + \mathbf{Z}\alpha_i. \tag{2.5.8}$$

Then

$$r_i^{(m_i,n_i)} r_i \lambda \in \lambda + \mathbf{Z}\alpha_i + \mathbf{Z}\delta_2.$$

This implies that

$$\begin{aligned} & r_1^{(m_1,n_1)} r_1 r_2^{(m_2,n_2)} r_2 \cdots r_\ell^{(m_\ell,n_\ell)} r_\ell(\lambda+\mu) \\ & \quad \in r_\ell^{(m_\ell,n_\ell)} r_\ell(\lambda+\mu) + \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \cdots + \mathbf{Z}\alpha_{\ell-1} + \mathbf{Z}\delta_2 \\ & \quad = \lambda + \mu + \nu_\ell(\lambda+\mu) + \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \cdots + \mathbf{Z}\alpha_{\ell-1} + \mathbf{Z}\delta_2 \end{aligned}$$

So, by (2.5.7)

$$w\lambda \equiv \lambda + \mu + \nu_\ell(\lambda+\mu), \qquad \text{mod } \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \cdots + \mathbf{Z}\alpha_{\ell-1} + \mathbf{Z}\delta_2. \tag{2.5.9}$$

Take $\lambda = \delta_2$. Since $w \in \dot{W}$, $w\delta_2 = \delta_2$. Then by (2.5.9),

$$\mu + \nu(\delta_2 + \mu) \in \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \cdots + \mathbf{Z}\alpha_{\ell-1} + \mathbf{Z}\delta_2,$$

which implies $m_\ell = k_\ell$. Suppose $\lambda_0 \in \Gamma$ satisfies the condition ($\Gamma 2$), i.e., $(\lambda_0 \mid \alpha_i) = 0, (\lambda_0 \mid \delta_2) = 1$. Then $w\lambda_0 = \lambda_0$. So $\mu + \nu_\ell(\lambda_0 + \mu) \in \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \cdots + \mathbf{Z}\alpha_{\ell-1} + \mathbf{Z}\delta_2$ and $n_\ell = 0$ (see (2.5.8)). Now (2.5.7) is reduced to

$$w\lambda \equiv r_1^{(m_1,n_1)} r_1 r_2^{(m_2,n_2)} r_2 \cdots r_{\ell-1}^{(m_{\ell-1},n_{\ell-1})} r_{\ell-1}(\lambda+\mu') \qquad \text{mod } \mathbf{Z}\delta_2.$$

where $\mu' = \mu - k_\ell\alpha_\ell$. Inductively for all $i, m_i = k_i, n_i = 0$. Finally, by (2.5.7) again, w fixes $\dot{Q}$ pointwise, and hence $w = 1$. □

LEMMA 2.5.8. *Let $\hat{n} \in \widehat{N}$ such that $n = \Phi(\hat{n}) : V^\lambda \to V^\lambda$, for all $\lambda \in \Gamma$. Then $\hat{n} \in \widehat{H}$. Moreover*

$$\hat{n} = \hat{z}\hat{h}_{\alpha_1}(c_1)\cdots\hat{h}_{\alpha_\ell}(c_\ell)$$

for some $\hat{z} \in \widehat{C}, c_i \in \mathbb{C}^\times$.

PROOF: Recall that $\widehat{N}/\widehat{H} \simeq \dot{W}$. Let w be the image of $\hat{n}$ in $\dot{W}$ under the natural map. Write $w = r_{\beta_1}\cdots r_{\beta_k}$ as a product of simple reflections of $\dot{W}$, i.e. $\beta_i \in \dot{\Pi}$. Then

$$\hat{n} = \hat{h}^{-1}\hat{n}_{\beta_1}(1)\cdots\hat{n}_{\beta_k}(1)$$

for some $\hat{h} \in \widehat{H}$. Then $\hat{h}V^\lambda = V^{w\lambda}$ for all $\lambda \in \Gamma$. By Lemmas 2.5.5 and 2.5.6, there exist $\hat{z} \in \widehat{C}, u_i = c_i s^{m_i} t^{n_i} \in R_2^\times, 1 \le i \le \ell$, such that

$$\hat{h} = \hat{z}\hat{h}_{\alpha_1}(u_1)\cdots\hat{h}_{\alpha_\ell}(u_\ell).$$

By Propositions 2.5.2 and 2.5.3,

$$w\lambda \equiv r_1^{(m_1,n_1)}r_1\cdots r_\ell^{(m_\ell,n_\ell)}r_\ell\lambda \qquad (\mathrm{mod}\ \mathbb{Z}\delta_2)$$

for all $\lambda \in \Gamma$. By Lemma 2.5.7, we have $m_i = n_i = 0, w = 1$. This implies $\hat{n} \in \widehat{H}$.

Moreover from the above proof, we have that

$$\hat{n} = \hat{z}\hat{h}_{\alpha_1}(c_1)\cdots\hat{h}_{\alpha_\ell}(c_\ell)$$

for some $\hat{z} \in \widehat{C}, c_i \in \mathbb{C}^\times$. □

THEOREM 2.5.9. *Suppose $\mathrm{rank}\,\dot{\Delta} \ge 2$. Then $ker\Phi \cap \widehat{N} = \widehat{C}_0$, the subgroup of $St(\dot{\Delta}, R_2)$ generated by Steinberg symbols $\{c, d\}\hat{}$ for all $c, d \in \mathbb{C}^\times$.*

PROOF: $ker\Phi \cap \widehat{N} \supset \widehat{C}_0$ follows from Proposition 2.5.3(iii). Conversely, let $\hat{n} \in ker\Phi \cap \widehat{N}$. Then for all $\lambda \in \Gamma$, $\Phi(\hat{n})V^\lambda = V^\lambda$. By Lemma 2.5.8,

$$\hat{n} = \hat{z}\hat{h}_{\alpha_1}(c_1)\cdots\hat{h}_{\alpha_\ell}(c_\ell)$$

for some $\hat{z} \in \widehat{C}, c_i \in \mathbb{C}^\times$. By (2.2.6), there exist $c, d \in \mathbb{C}^\times, m \in \mathbb{Z}$, such that

$$\hat{z}\{c, s\hat{\}}\{d, t\hat{\}}\{s, t\hat{\}}^m \in \widehat{C}_0.$$

We have

$$\Phi\big(\{c^{-1}, s\hat{\}}\{d^{-1}, t\hat{\}}\{s, t\hat{\}}^{-m}\hat{h}_{\alpha_1}(c_1)\cdots\hat{h}_{\alpha_\ell}(c_\ell)\big) = 1 \in G_\Gamma.$$

By Proposition 2.5.3, only $\{s, t\hat{\}}$ acts as a translation mapping V^λ to $V^{\lambda+\delta_2}$, the others act as scalar maps on V^λ. Then $m = 0$. So for all $\lambda \in \Gamma, y \in S(\mathfrak{b}_-)$, $\Phi(\hat{n})$ also acts on $e^\lambda \otimes y$ as a scalar multiplication by

$$cd^{(\delta_2|\lambda)}c_1^{(\alpha_1|\lambda)}\cdots c_\ell^{(\alpha_\ell|\lambda)},$$

hence

$$cd^{(\delta_2|\lambda)}c_1^{(\alpha_1|\lambda)}\cdots c_\ell^{(\alpha_\ell|\lambda)} = 1$$

for all $\lambda \in \Gamma$. By (Γ1) and (Γ2), we get that $c = d = c_1 = \cdots = c_\ell = 1$. Then $\hat{z} \in \widehat{C}_0$, and hence $\hat{n} \in \widehat{C}_0$. □

PROPOSITION 2.5.10. *Suppose* $\operatorname{rank}\dot{\Delta} \geq 2$. *Let* $\widetilde{H}$ *be as in* §2.2. *Then*

(i) $\widetilde{H} \lhd \widetilde{N} = \widehat{N}$.

(ii) There is an isomorphism $\tilde{\pi}$ *from* $\widetilde{N}/\widetilde{H}$ *onto the toroidal Weyl group* W, *such that* $\tilde{\pi}(\tilde{n}_\mu(c)) = r^{(m)}_{\alpha+n\delta_2}$ *where* $\mu = \alpha + m\delta_1 + n\delta_2 \in {}^{re}\Delta$.

PROOF: (i) follows from (R5′). Now recall that $\tilde{n}_\nu(c) = \hat{n}_\alpha(cs^m t^n)$(Lemma 2.2.4) and for any $\lambda \in \Gamma$, $\tilde{n}_\mu(c)V^\lambda = V^{r^{(m)}_{\alpha+n\delta_2}\lambda}$ (Proposition 2.5.2). If any element $\tilde{n} \in \widetilde{N}$ is expressed as $\tilde{n}_{\mu_1}(c_1)\cdots\tilde{n}_{\mu_k}(c_k)$, then define

$$\pi(\tilde{n}) = r^{(m_1)}_{\beta_1+n_1\delta_2}\cdots r^{(m_k)}_{\beta_k+n_k\delta_2} \tag{2.5.10}$$

where $\mu_i = \beta_i + m_i\delta_1 + n_i\delta_2 \in {}^{re}\Delta$. This is a well defined homomorphism from $\widetilde{N}$ to W. Indeed, if $\tilde{n} = \tilde{n}_{\nu_1}(c_1')\cdots\tilde{n}_{\nu_l}(c_l')$, then $r^{(p_1)}_{\gamma_1+q_1\delta_2}\cdots r^{(p_l)}_{\gamma_k+q_l\delta_2}$ ($\nu_i = \gamma_i + p_i\delta_1 + q_i\delta_2 \in {}^{re}\Delta$) is equal to $r^{(m_1)}_{\beta_1+n_1\delta_2}\cdots r^{(m_k)}_{\beta_k+n_k\delta_2}$ by looking at the action on V^λ. So $\Phi(\hat{n})V^\lambda = V^{\pi(\tilde{n})\lambda}$ for all λ and π is surjective. It remains to show $\widetilde{H} = ker(\pi)$.

By Proposition 2.5.3(i) and Lemma 2.2.4, $\tilde{h}_\mu(c)$ maps V^λ to V^λ, so $\pi(\tilde{h}) = 1$ for all $\tilde{h} \in \widetilde{H}$. This proves $\widetilde{H} \subset ker(\pi)$. Conversely, let $\tilde{n} \in ker(\pi)$. Then $\tilde{n}V^\lambda = V^\lambda$ for all $\lambda \in \Gamma$. By Lemma 2.5.8,

$$\tilde{n} = \hat{z}\hat{h}_{\alpha_1}(c_1)\cdots\hat{h}_{\alpha_\ell}(c_\ell)$$

for some $\hat{z} \in \widehat{C}, c_i \in \mathbb{C}^\times$. Let $\hat{z} = \hat{z}_0\hat{\{c,s\}}\hat{\{d,t\}}\hat{\{s,t\}}^m$ for $\hat{z}_0 \in \widehat{C}_0, c, d \in \mathbb{C}^\times, m \in \mathbb{Z}$. Then $m = 0$ as in the proof of Theorem 2.5.9. Then by Lemma 2.2.6,

$$\tilde{n} = \hat{z}_0\hat{\{c,s\}}\hat{\{d,t\}}\hat{h}_{\alpha_1}(c_1)\cdots\hat{h}_{\alpha_\ell}(c_\ell) \in \widetilde{H}.$$

Since $\widehat{C}_0 \subset \widetilde{H}$, then $\widetilde{H} = ker(\pi)$. □

By Lemma 2.5.1, $W \simeq W_a \ltimes Q_a$, where W_a is the affine Weyl group and $Q_a = \dot{Q} + \mathbb{Z}\delta_2$ acts on Γ as translations $t_q, q \in Q_a$. Let f be the map from W onto W_a obtained from $W \simeq W_a \ltimes Q_a$. Let $\pi_a = f \cdot \pi$ where $\pi : \widehat{N} \to W$ is defined as in (2.5.10). Thus π_a is a group homomprphism from $\widehat{N}$ onto W_a. Clearly, we have by definition,

LEMMA 2.5.11. *Suppose rank $\dot{\Delta} \geq 2$. Let $\hat{n} \in \widehat{N}$. Then $\hat{n} \in ker(\pi_a)$ if and only if there exists an element $\mu \in Q_a$ such that*

$$\Phi(\hat{n}) : V^\lambda \to V^{\lambda+\mu},$$

for all $\lambda \in \Gamma$. □

Let

$$\widehat{H}_s := \widehat{C} \cdot \langle \hat{h}_\alpha(cs^m) \mid \alpha \in \dot{\Delta}, c \in \mathbb{C}^\times, m \in \mathbb{Z} \rangle$$

Then by (R5), $\widehat{H}_s \lhd \widehat{N}$. We show $ker(\pi_a) = \widehat{H}_s$.

Recall that $\widehat{N}/\widehat{H} \simeq \dot{W}$. Let $\hat{n} \in ker(\pi_a)$ and $w \in \dot{W}$ correspond to $\hat{n}$. Write $w = r_{\beta_1}\cdots r_{\beta_k}$ as a product of simple reflections of $\dot{W}$, i.e. $\beta_i \in \dot{\Pi}$. Then

$$\hat{n} = \hat{h}^{-1}\hat{n}_{\beta_1}(1)\cdots\hat{n}_{\beta_k}(1)$$

for some $\hat{h} \in \widehat{H}$. Then by Lemma 2.5.11, there is $\mu = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_\ell\alpha_\ell + k_0\delta_2 \in Q_a$ such that

$$\hat{h}V^{\lambda+\mu} = V^{w\lambda}$$

for all $\lambda \in \Gamma$. By Lemmas 2.5.5 and 2.5.6, there exist $\hat{z} \in \widehat{C}, u_i = c_i s^{m_i} t^{n_i} \in R_2^\times, 1 \le i \le \ell$, such that

$$\hat{h} = \hat{z}\hat{h}_{\alpha_1}(u_1)\cdots\hat{h}_{\alpha_\ell}(u_\ell).$$

By Propositions 2.5.2 and 2.5.3,

$$w\lambda \equiv r_1^{(m_1,n_1)}r_1 \cdots r_\ell^{(m_\ell,n_\ell)}r_\ell(\lambda+\mu) \pmod{\mathbb{Z}\delta_2}$$

for all $\lambda \in \Gamma$. By Lemma 2.5.7, we have $m_i = k_i, n_i = 0, 1 \le i \le \ell$ and $w = 1$. This implies $\hat{n} \in \widehat{H}$. We may assume that

$$\hat{n} = \hat{z}\hat{h}_{\alpha_1}(c_1 s^{k_1})\cdots\hat{h}_{\alpha_\ell}(c_\ell s^{k_\ell})$$

for some $\hat{z} \in \widehat{C}, c_i \in \mathbb{C}^\times$. Then $\hat{n} \in \widehat{H}_s$.

Conversely, applying the proof of Lemma 2.5.5, we get that any element of $\widehat{C}\widehat{H}_s$ can be written as

$$\hat{z}\hat{h}_{\alpha_1}(c_1 s^{k_1})\cdots\hat{h}_{\alpha_\ell}(c_\ell s^{k_\ell})$$

for $\hat{z} \in \widehat{C}, c_i \in \mathbb{C}^\times, k_i \in \mathbb{Z}$. Such an element sends V^λ to $V^{\lambda+\sum_{i=1}^{\ell} k_i\alpha_i + k_0\delta_2}$, where $k_0 \in \mathbb{Z}$ is independent of λ. So, $\widehat{H}_s \subset ker(\pi_a)$ by Lemma 2.5.11. This proves

PROPOSITION 2.5.12. *Suppose* $\operatorname{rank}\dot{\Delta} \ge 2$.

$$\widehat{N}/\widehat{H}_s \simeq \widehat{N}/\widehat{H}_t \simeq W_a$$

where $\widehat{H}_t := \widehat{C}\cdot\langle\hat{h}_\alpha(ct^m) \mid \alpha \in \dot{\Delta}, c \in \mathbb{C}^\times, m \in \mathbb{Z}\rangle$. □

Since calculations in the proofs of Proposition 2.5.10 and Proposition 2.5.12 are based on the vertex representation V_Γ, similar results hold in the group G_Γ.

COROLLARY 2.5.13. *Suppose rank* $\dot{\Delta} \geq 2$. *Let* Φ *be the map from* $St(\dot{\Delta}, R_2)$ *onto* G_Γ. *Then*

(i) $\Phi(\widetilde{N})/\Phi(\widetilde{H}) \simeq W$.

(ii) $\Phi(\widehat{N})/\Phi(\widehat{H}_s) \simeq \Phi(\widehat{N})/\Phi(\widehat{H}_t) \simeq W_a$. □

Since the toroidal Lie algebra $\mathfrak{t}$ contains two visible affine Lie algebras with respect to two varibles s and t, we will show that attached groups have similar relationship. See [KP], [MP](§6.1) and [Ti1] for more details about constructions of affine Kac-Moody groups from integrable representions of respective affine Lie algebras.

For an affine Lie algebras $\hat{\mathfrak{g}}$ of type $X_\ell^{(1)}(X = A, D, \text{or } E)$, denote by G_{KM} the Kac-Moody Group of respective type. Then G_{KM} has a presentation as follows:

generators : $\quad \bar{x}_\mu(a), \quad \mu \in \dot{\Delta} + \mathbb{Z}\delta, a \in \mathbb{C}$.

relations:

(R1′) $\quad \bar{x}_\mu(a)\bar{x}_\mu(b) = \bar{x}_\mu(a+b)$

(R2′) $\quad (\bar{x}_\mu(a), \bar{x}_\nu(b)) = \begin{cases} 1, & \text{if } \mu+\nu \notin (\dot{\Delta}+\mathbb{Z}\delta)\cup\{0\} \\ \bar{x}_{\mu+\nu}(\varepsilon(\mu,\nu)ab), & \text{if } (\mu \mid \nu) = -1\ , \end{cases}$

(R0′) $\quad \bar{h}_\mu(c)\bar{h}_\mu(d) = \bar{h}_\mu(cd)$

where $\mu, \nu \in \dot{\Delta} + \mathbb{Z}\delta$, $a, b \in \mathbb{C}, c, d \in \mathbb{C}^\times$, and δ is the null root of $\hat{\mathfrak{g}}$, and elements $\bar{h}_\mu(c)$ are defined in a similar way as before.

Let G_s and G_t be the subgroups of the universal toroidal group $\overline{G}$ generated by $\{\bar{x}_\mu(a) \mid \mu \in a \in \mathbb{C}, \dot{\Delta} + \mathbb{Z}\delta_1\}$ and $\{\bar{x}_\mu(a) \mid a \in \mathbb{C}, \mu \in \dot{\Delta} + \mathbb{Z}\delta_2\}$ respectively. Then the map f from G_{KM} into G_t defined by

$$f : \bar{x}_\mu(a) \mapsto \bar{x}_\mu(a) \tag{2.5.11}$$

is a well defined epimorphism from Lemma 2.3.4 and Corollary 2.3.7. Indeed, we have

PROPOSITION 2.5.14. *The map f defined in (2.5.11) is an isomorphism. So $G_{KM} \simeq G_t \simeq G_s$.*

PROOF: If $f(x) = 1 \in G_t$ for $x \in G_{KM}$, then for any integrable representation of $\mathfrak{t}$, $f(x)$ acts as the identity map. To show $x = 1$ in G_{KM}, it suffices to show that x acts as the identity map on any integrable representation of $\hat{\mathfrak{g}}$.

We identify the affine Lie algebra $\hat{\mathfrak{g}}$ with $\mathbb{C}[s, s^{-1}] \otimes \dot{\mathfrak{g}} + \mathbb{C}\overline{s^{-1}ds}$. Define a linear map $\pi : \mathfrak{t} \to \hat{\mathfrak{g}}$ given by

$$\pi(g \otimes x) = g(s,1) \otimes x \qquad \pi\overline{(dg)g'} = \overline{(dg(s,1))g'(s,1)}$$

for $x \in \dot{\mathfrak{g}}$, $g = g(s,t), g' = g'(s,t) \in R_2$. Then π is a homomorphism of Lie algebras.

Let (ρ, M) be any integrable representation of $\hat{g}$. Then $(\tilde{\rho}, M)$ is a representation of $\mathfrak{t}$ where $\tilde{\rho} = \rho\pi$. This gives a representation $\tilde{\boldsymbol{\rho}}$ of $\overline{G}$ on M such that (2.3.2) holds. Similarly, ρ lifts to a representation $\boldsymbol{\rho}$ of G_{KM} on M, such that

$$\exp\rho(X) = \boldsymbol{\rho}\exp X$$

where $X \in \mathfrak{s}^n \otimes \dot{\mathfrak{g}} \subset \hat{\mathfrak{g}}, n \in \mathbb{Z}$. Then we have a commutative diagram

$$\begin{array}{ccc} G_{KM} & \xrightarrow{f} & \overline{G} \\ & \searrow \boldsymbol{\rho} \qquad \tilde{\boldsymbol{\rho}} \swarrow & \\ & GL(M) & \end{array}$$

In fact , for all $X \in \mathfrak{s}^n \otimes \dot{\mathfrak{g}}$,

$$\begin{aligned} \tilde{\boldsymbol{\rho}}(f(\exp X)) &= \tilde{\boldsymbol{\rho}}(\exp X) \\ &= \exp\tilde{\rho}(X) \\ &= \exp\rho(X) \\ &= \boldsymbol{\rho}\exp x \end{aligned}$$

Since all such $\exp x$'s are generators of G_{KM}, so $\tilde{\boldsymbol{\rho}} f = \boldsymbol{\rho}$.

Now if $x \neq 1$ in G_{KM}, then there is an integrable representation (ρ, M) of $\hat{\mathfrak{g}}$ such that $\rho(x) \neq 1 \in GL(M)$. But $\rho(x) = \tilde{\rho} f(x) = 1$ is a contradiction. So $x = 1$, that is , f is injective and an isomorphism. □

This result can also be easily proved by using the vertex representation of $\mathfrak{t}$ and the fact that G_{KM} has a Bruhat decomposition. More specifically, the vertex representation space V_Γ of $\mathfrak{t}$ also affords a faithful representation for G_{KM}.

§2.6. A Proof of Conjecture 2.3.8 in the $A_\ell, \ell \geq 4$, Cases.

In this section, we assume $\dot{\Delta}$ is of type A_ℓ, $\ell \geq 4$. Often we write $n = \ell + 1$.

With the help of the vertex representation V_Γ of $\mathfrak{t}$, we shall show the conjecture in our cases. The lattice Γ is still assumed to satisfy conditions ($\Gamma 1$) and ($\Gamma 2$).

The approach to the problem is to find the kernel of the map $\Phi : St(\dot{\Delta}, R_2) \to G_\Gamma$. This is connected to K-theory, specially, the K_2-functor. We only state some results in K-theory which are useful for us. One can find more details in [HO] and [Mi].

In our cases, $\dot{\mathfrak{g}} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$. So $R_2 \otimes \dot{\mathfrak{g}} = \mathfrak{sl}_{\ell+1}(R_2)$. There is a standard one to one correspondence between $\dot{\Delta}$ and the set $\{(ij) \mid i \neq j\}$. So a pair (ij) will be used to denote a root in $\dot{\Delta}$ and both notations will be used. Let E_{ij} be the $(\ell+1) \times (\ell+1)$ matrix with 1 at the ij-th position, 0 elsewhere. Then the set $\{E_{ij}, -E_{ji} \mid i < j\}$ generates a Chevalley basis of $\dot{\mathfrak{g}}$ in the sence of Bourbaki (cf. Definition 2.1.1). Let $E_{\ell+1}(R_2)$ be the elementary group generated by all elementary matrices $\{I + aE_{ij} \mid 1 \leq i \neq j \leq \ell+1, a \in R_2\}$, where $I := I_{\ell+1,\ell+1}$ is the identity matrix. Let $St_{\ell+1}(R_2)$ denote the Steinberg group $St(\dot{\Delta}, R_2)$ associated to the above Chevalley basis (see Remark 2.1.2). Then there is a natural epimorphism $\Psi : St_{\ell+1}(R_2) \to E_{\ell+1}(R_2)$ given by

$$\Psi(\hat{x}_{ij}(a)) := \begin{cases} 1 + aE_{ij}, & i < j \\ 1 - aE_{ij}, & i > j. \end{cases}$$

The kernel of Ψ is denoted by $K_{2,\ell+1}(R_2)$.

LEMMA 2.6.1. *$\Psi : St_{\ell+1}(R_2) \to E_{\ell+1}(R_2), \ell \geq 4$, is a universal central extension.*

PROOF: See 4.2.20 of [HO]. □

The next lemma is the main reason why we restrict ourself to the cases $\dot{\Delta} = A_\ell$. It is due to Hurrelbrink [Hu].

LEMMA 2.6.2. *$K_{2,\ell+1}(R_2), \ell \geq 4$, is generated by Steinberg symbols $\{c, d\hat{}\}, \{c, s\hat{}\}$, $\{c, t\hat{}\}$, $\{s, t\hat{}\}$ for all $c, d \in \mathbb{C}^\times$.* □

LEMMA 2.6.3. *Let $n \geq 2$. Then*

$$Cent(E_n(R_2)) = \{\zeta I \mid \zeta \text{ is an } n\text{-th root of unity}\}$$

PROOF: By 1.2.14 of [HO], the centralizer of $E_n(R_2)$ in $GL_n(R_2)$ is the set $\{uI \mid u \in R_2^\times\}$. If $x \in Cent(E_n(R_2))$, then $x = uI$ for some $u \in R_2^\times$. Also $E_n(R_2) \subset SL_n(R_2)$, $det(x) = 1$ and so $u^n = 1$. Since $R_2^\times = \{cs^m t^n \mid c \in \mathbb{C}^\times, m, n \in \mathbb{Z}\}$, we see that $u \in \mathbb{C}$ and u is an n-th root of unity.

On the other hand, if ζ is an n-th root of unity, then ζI is in $SL_n(\mathbb{C})$, hence in $E_n(\mathbb{C})$ since $SL_n(\mathbb{C}) = E_n(\mathbb{C})$. So the result follows. □

LEMMA 2.6.4. *Let $n \geq 5$. Then $K_2' := \Psi^{-1}(Cent(E_n(R_2))) \subset Cent(St_n(R_2))$, and hence K_2' is abelian.*

PROOF: Suppose $x \in K_2'$. Then $\Psi(x) \in Cent(E_n(R_2))$. Let f be the inner automorphism of $St_n(R_2)$ defined by $f(y) := xyx^{-1}$ for all $y \in St_n(R_2)$. Now the following diagram commutes.

$$\begin{array}{ccc} St_n(R_2) & \xrightarrow{f} & St_n(R_2) \\ & \searrow^{\Psi} \quad \swarrow_{\Psi} & \\ & E_n(R_2) & \end{array}$$

Since Ψ is a universal central extension, f is equal to the identity map. Thus $y = xyx^{-1}$ for all $y \in St_n(R_2)$. So $x \in Cent(St_n(R_2))$. □

LEMMA 2.6.5. *Let $n \geq 5$. If ζ is an n-th root of unity, then there exists an element $h \in \widehat{H}_0$ such that $\Psi(h) = \zeta I$, where $\widehat{H}_0$ is the subgroup of $St_n(R_2)$ generated by all $\hat{h}_\alpha(c), \alpha \in \dot{\Delta}, c \in \mathbb{C}^\times$. Futhermore $h \in K_2'$.*

PROOF: We consider $St_n(\mathbb{C})$ first. By Corollary 1(b) (P.40) of [St2], $Cent(St_n(\mathbb{C})) \subset \widehat{H}_0$ (thought of as the one contained in $St_n(\mathbb{C})$). Since the canonical map $St_n(\mathbb{C}) \to SL_n(\mathbb{C}) = E_n(\mathbb{C})$ is surjective, the matrix $\zeta I \in SL_n(\mathbb{C})$ has a preimage,

say h', in $St_n(\mathbb{C})$. Also this map is a universal central extension, so by the proof of Lemma 2.6.4, $h' \in wH_0$. We may write h' as a product of $\hat{h}_\alpha(c)$'s.

Let τ be the map from $St_n(\mathbb{C})$ into $St_n(R_2)$ defined by $\tau(\hat{x}_\alpha(c)) = \hat{x}_\alpha(c)$. So τ maps $\widehat{H}_0(\subset St_n(\mathbb{C}))$ onto $\widehat{H}_0(\subset St_n(R_2))$. Let $h = \tau(h')$. Our result follows from the commutative diagram

$$\begin{array}{ccc} St_n(\mathbb{C}) & \longrightarrow & E_n(\mathbb{C}) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ St_n(R_2) & \xrightarrow{\Psi} & E_n(R_2) \end{array}$$

□

LEMMA 2.6.6. *Let ζ be a primitive n-th root of unity and $h \in \widehat{H}_0$ be chosen as in Lemma 2.6.5 associated with ζ. Then*

$$K_2' = \cup_{i=0}^{n-1} h^i K_{2,n}(R_2).$$

PROOF: The result follows clearly from facts that $\Psi|_{K_2'}$ maps K_2' onto $Cent(E_n(R_2))$ $= \{\zeta^i I \mid 0 \le i \le n-1\}$ with the kernel $K_{2,n}(R_2)$ and h is a preimage of a generator of $Cent(E_n(R_2))$ □

Now we consider the vertex representation $(\dot{\rho}, V_\Gamma)$ of $\mathfrak{t}$, where Γ is as before. We will wiew elements of $\mathfrak{t}$ as operators on V_Γ, that is, $\mathfrak{t} \subset \mathrm{End}(V)$. Also $G_\Gamma \subset \mathrm{Aut}(V)$.

For any $X \in \mathfrak{t}, g \in G_\Gamma$, define a linear map in $\mathrm{End}(V_\Gamma)$, denoted by $\mathrm{Ad}g.X$ by

$$(\mathrm{Ad}g.X)v := g.X.g^{-1}v \tag{2.6.1}$$

for all $\in V_\Gamma$. If $g = \exp Y$ ($\exp Y$ really means $\exp\rho(Y)$) for some $Y \in \mathfrak{t}^\mu, \mu \in {}^{re}\Delta$, then

$$\exp Y.X.\exp(-Y).v = ((\exp adY)X).v \tag{2.6.2}$$

by Proposition 6.1.2 of [MP]. Then $\mathrm{Ad}g.X = g.X.g^{-1} \in \mathfrak{t}$. Since G_Γ is generated by all $\exp \mathfrak{t}^\mu, \mu \in {}^{re}\Delta$, then for any $g \in G_\Gamma$, $g.X.g^{-1} \in \mathfrak{t} \subset \mathrm{End}(V_\Gamma)$. $\mathrm{Ad}g$ is a linear operator on the vector space $\mathfrak{t}$. Note that

$$(\mathrm{Ad}g_1g_2.X) = g_1.g_2.X.g_2^{-1}.g_1^{-1}.v$$

and

$$(\mathrm{Ad}g.[X,Y]).v = g.(X.Y - Y.X).g^{-1}.v$$

$$= [\mathrm{Ad}g.X, \mathrm{Ad}g.Y].v$$

Then Ad maps G_Γ into the automorphism group $\mathrm{Aut}(\mathfrak{t})$ of $\mathfrak{t}$ as a Lie algebra. Let

$$\mathrm{Int}(\mathfrak{t}) := \langle \exp ad\mathfrak{t}^\mu \mid \mu \in {}^{re}\Delta \rangle$$

be the group of inner automorphisms of $\mathfrak{t}$. so by (2.6.2) we have a surjective homomorphism

$$\mathrm{Ad}G_\Gamma \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Int}(\mathfrak{t}). \tag{2.6.3}$$

Let $\mathrm{Int}(\mathfrak{sl}_n(R_2))$ be the subgroups of $\mathrm{Aut}(\mathfrak{sl}_n(R_2))$ generated by $\exp ad(aE_{ij})$ for all $a \in R_2$ and $1 \le i \ne j \le n$. Note that for $X \in \mathfrak{sl}_n(R_2)$

$$\exp ad(aE_{ij}).X = (I + aE_{ij}) \cdot X \cdot (I + aE_{ij})^{-1}$$

where "$\cdot$" means the multiplication of matrices. It is easy to check that the map from $E_(R_2)$ to $\mathrm{Int}(\mathfrak{sl}_n(R_2))$ given by

$$I + aE_{ij} \longmapsto \exp ad(aE_{ij})$$

is a well defined homomorphism. Moreover, it is surjective and has the center of $E_n(R_2)$ as the kernel (see the proof of Lemma 2.6.3). Denote the center of $E_n(R_2)$ by Z. Then

$$\mathrm{Int}(\mathfrak{sl}_n(R_2)) \simeq E_n(R_2)/Z. \tag{2.6.4}$$

As a consequence of $\mathfrak{t}$ being a (universal) central extension of $\mathfrak{sl}_n(R_2) = R_2 \otimes \dot{\mathfrak{g}}$, we have

LEMMA 2.6.7.

$$Int(\mathfrak{t}) \simeq Int(\mathfrak{sl}_n(R_2)) \simeq E_n(R_2)/Z,$$

where $Z = Cent(E_n(R_2))$.

PROOF: Let $y \in \mathrm{Int}(\mathfrak{t})$. Since y acts on the center Ω/R_2 of $\mathfrak{t}$ trivially, there is a unique map $\bar{y} \in \mathrm{End}(R_2 \otimes \dot{\mathfrak{g}})$ such that the diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{y} & \mathfrak{t} \\ {\scriptstyle nat}\downarrow & & \downarrow{\scriptstyle nat} \\ R_2 \otimes \dot{\mathfrak{g}} & \xrightarrow{\bar{y}} & R_2 \otimes \dot{\mathfrak{g}} \end{array}$$

commutes. Now y can be expressed as a product of elements in $\exp ad t^{\mu}$. It is clear that $\bar{y} \in \mathrm{Int}(R_2 \otimes \dot{\mathfrak{g}})$ and $y \mapsto \bar{y}$ defines a homomorphism from $\mathrm{Int}(\mathfrak{t})$ into $\mathrm{Int}(R_2 \otimes \dot{\mathfrak{g}})$. Conversely, since $R_2 \otimes \dot{\mathfrak{g}}$ is a vector subspace (not a subalgebra) of $\mathfrak{t}$, for each $\bar{y} \in \mathrm{Int}(R_2 \otimes \dot{\mathfrak{g}})$,

$$\begin{aligned} y(X) &:= \bar{y}(X) \quad \text{if } X \in R_2 \otimes \dot{\mathfrak{g}} \\ y(z) &:= z \qquad \text{if } z \in \Omega/R_2 \end{aligned}$$

defines an endomorphism $\tilde{x}$ of $\mathfrak{t}$. Indeed, $\tilde{x} \in \mathrm{Int}(\mathfrak{t})$, which can be checked for generators $\exp adY$, $Y \in R_2 \otimes \dot{\mathfrak{g}}^{\alpha}$. Then

$$y \longleftrightarrow \bar{y}$$

defines a isomorphism between $\mathrm{Int}(\mathfrak{t})$ and $\mathrm{Int}(R_2 \otimes \dot{\mathfrak{g}})$. Thus the lemma holds by (2.6.4). □

In summary, we have

PROPOSITION 2.6.8. *There is a homomorphism, also denoted by Ad, of groups from* G_Γ *onto* $E_n(R_2)/Z$ *such that the diagram*

$$\begin{array}{ccc} St_n(R_2) & \xrightarrow{\Psi} & E_n(R_2) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle nat} \\ G_\Gamma & \xrightarrow{\mathrm{Ad}} & \overline{E}_n(R_2)/Z \end{array}$$

commutes.

PROOF: It remains to show the diagram commutes. It is enough to show that on the generators. This is clear since

$$\mathrm{Ad}.\exp X = \exp adX$$

for $X \in \mathfrak{t}^{\mu}, \mu \in {}^{re}\Delta$. □

REMARK 2.6.9. *The above proposition holds for groups arising from any faithful and integrable representation of* $\mathfrak{t}$.

LEMMA 2.6.10. $ker(\Phi) = \widehat{C}_0$.

PROOF: Let Ψ' be the composition map from $St_n(R_2)$ to $E_n(R_2)/Z$ in Proposition 2.6.8. Then by Proposition 2.6.8,

$$ker(\Phi) \subset ker(\Psi') = \Psi^{-1}(Z) = K_2'.$$

Now $K_2' = \cup_{i=0}^{n-1} h^i K_{2,n}(R_2)$ (Lemma 2.6.6) for some $h \in \widehat{H}_0$. Lemma 2.6.2 implies $K_{2,n}(R_2) \subset \widehat{N}$, hence $K_2' \subset \widehat{N}$ and $ker(\Phi) \subset \widehat{N}$. By Theorem 2.5.9, $ker(\Phi) = \widehat{C}_0$. □

THEOREM 2.6.11.

(i) The relations (R0′), (R1′) and (R2′) for generators $\{x_{\mu}(c) \mid \mu \in {}^{re}\Delta, c \in \mathbb{C}\}$ *of* G_{Γ}, *are the defining relations of* G_{Γ}.

(ii) The relations (R0′), (R1′) and (R2′) for generators $\{\bar{x}_{\mu}(c) \mid \mu \in {}^{re}\Delta, c \in \mathbb{C}\}$ *of* $\overline{G}$, *are the defining relations of* $\overline{G}$.

(iii) $\overline{G} \simeq G_{\Gamma}$.

PROOF: (i) is a consequence of Lemma 2.6.10. (ii) and (iii) follow from (i) and the

following commutative diagram

$$\begin{array}{ccc} St(^{re}\Delta, \mathbb{C}) & \xrightarrow{\Phi} & G_\Gamma \\ & \searrow \phi \quad \psi \nearrow & \\ & \overline{G} & \end{array}$$

where all the maps are as before. □

This theorem shows Conjecture 2.3.8 in the cases when $\dot{\Delta}$ is of type A_ℓ for $\ell \geq 4$.

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Chapter 3. Toroidal Weyl Groups

§3.0. Introduction.

In [MRY] and [EMY], Moody, Eswara Rao and Yokonuma studied the toroidal Lie algebras and their representations. An n-toroidal Lie algebra is a universal central extension of the Lie algebra $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \otimes \dot{\mathfrak{g}}$, where $\dot{\mathfrak{g}}$ is the finite dimensional simple Lie algebra over the complex field $\mathbb{C}$. The terminology "toroidal" comes about because the Lie algebra $\mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \otimes \dot{\mathfrak{g}}$ can be viewed as the Lie algebra of all polynomial maps from the complex torus $\mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ (n copies) to $\dot{\mathfrak{g}}$. When $n = 1$, the theory is nothing but the well known theory affine Lie algebras. In the theory of affine Lie algebras, affine Weyl groups is one of most important characters. Here we study the Weyl groups of n-toroidal Lie algebras, for general n.

Let $\dot{\Delta}$ be a finite root system of $\dot{\mathfrak{g}}$ of type A_ℓ, D_ℓ or E_ℓ and $\dot{Q}$ the finite root lattice of $\dot{\Delta}$. Under the standard normalized bilinear form $(\cdot \mid \cdot)$, $\dot{\Delta}$ is the subset of vectors with square length 2 of $\dot{Q}$. Extend $\dot{Q}$ to a *toroidal root lattice* $Q := Q^{[n]} := \dot{Q} \oplus \bigoplus_{1 \le i \le n} \mathbb{Z}\delta_i$. Then Q inherits a symmetric bilinear form from $\dot{Q}$ with the null roots δ_i in its radical. We extend Q to a lattice Γ on which $(\cdot \mid \cdot)$ is non-singular. The *n-toroidal Weyl group* $W = W^{[n]}$ is the group generated by all reflections of Γ in the vectors of Q with square length 2.

The affine Weyl group $W^{[1]}$ is isomorphic to $\dot{W} \ltimes \dot{Q}$, where elements in $\dot{Q}$ are realized as affine translations (see, for instance, [Ka]). Hence $\dot{Q}$ is an abelian subgroup of $W^{[1]}$. As the Weyl group of the affine Lie algebra $\mathbb{C}[s^{\pm 1}] \otimes \dot{\mathfrak{g}} \oplus \mathbb{C}\mathfrak{k}$, $W^{[1]}$ is a Coxeter group. In this paper, we will generalize these results of the case $n = 1$ as follows.

In §3.1, we show that a toroidal Weyl group is a semidirect product of finite Weyl group and a discrete Heisenberg group (a step 2 nilpotent group, instead of an abelian group) with a center which is a free abelian group of rank $n(n-1)/2$. In §3.2, we see that any toroidal Weyl group is a homomorphic image of some

indefinite Kac-Moody Weyl group and describe the kernel. In the final section, we show how toriodal Weyl groups constructed from lattices naturally arise from toroidal Lie algebras.

§3.1. Toroidal Weyl Groups.

Let $\dot{\Delta}$ be an irreducible reduced root system of type $X_\ell = A_\ell$, D_ℓ or E_ℓ. Fix a basis $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$ for $\dot{\Delta}$. Then the relative Cartan matrix $\dot{A}$ defines a symmetric nondegenerate bilinear form on the root lattice $\dot{Q} = \bigoplus_{i=1}^{\ell} \mathbf{Z}\,\alpha_i$. Let θ be the unique highest root of $\dot{\Delta}$ relative to $\dot{\Pi}$.

Fix an integer $n \in \mathbf{Z}_{>0}$. We will extend the root lattice to an $\ell+2n$ dimensional geometric lattice.

Let

$$\dot{I} := \{1, \ldots, \ell\}, \qquad J = \{\ell+1, \ell+2, \ldots, \ell+n\}$$

$$I_k := \dot{I} \cup \{k\} \qquad \text{for} \quad \ell+1 \le k \le \ell+n$$

$$I := \{1, \ldots, \ell, \ell+1, \ldots, \ell+n\}.$$

These index sets will be used all the time.

Take symbols δ_k, Λ_k for $k \in J$ and the free abelian group

$$\Gamma := \Gamma^{[n]} := \mathbf{Z}\alpha_1 \oplus \cdots \oplus \mathbf{Z}\alpha_\ell \oplus \mathbf{Z}\delta_{\ell+1} \oplus \cdots \oplus \mathbf{Z}\delta_{\ell+n} \oplus \mathbf{Z}\Lambda_{\ell+1} \oplus \cdots \oplus \mathbf{Z}\Lambda_{\ell+n}.$$

Then $\dot{Q}$ is a subgroup of Γ. Extend the bilinear form on $\dot{Q}$ to Γ as follows:

$(\cdot\,|\,\cdot) : \Gamma \times \Gamma \to \mathbf{Z}$ is a symmetric bilinear form which extends the form on $\dot{Q}$ and satisfies for $i \in \dot{I}$, $k, k' \in J$.

$$(\alpha_i \,|\, \delta_k) = (\alpha_i \,|\, \Lambda_k) = 0$$

$$(\delta_k \,|\, \delta_{k'}) = (\Lambda_k \,|\, \Lambda_{k'}) = 0$$

$$(\delta_k \,|\, \Lambda_{k'}) = \delta_{kk'}$$

The elements δ_k are called **null roots.**

The matrix of the extended form is

$$\begin{pmatrix} (\alpha_i\,|\,\alpha_j) & (\alpha_i\,|\,\delta_k) & (\alpha_i\,|\,\Lambda_k) \\ (\delta_k\,|\,\alpha_j) & (\delta_k\,|\,\delta_{k'}) & (\delta_k\,|\,\Lambda_{k'}) \\ (\Lambda_k\,|\,\alpha_j) & (\Lambda_k\,|\,\delta_{k'}) & (\Lambda_k\,|\,\Lambda_{k'}) \end{pmatrix} = \begin{pmatrix} \dot{A}_{\ell\times\ell} & 0 & 0 \\ 0 & 0 & I_{n\times n} \\ 0 & I_{n\times n} & 0 \end{pmatrix}_{(\ell+n)\times(\ell+n)} \tag{3.1.1}$$

It is clear that this bilinear form is nondegenerate. Let $\Delta := \dot{\Delta} + \mathbf{Z}\delta_{\ell+1} + \cdots + \mathbf{Z}\delta_{\ell+n}$. A typical element in Δ is

$$\mu = \alpha + \sum_{k\in J} m_k\delta_k, \qquad \alpha \in \dot{\Delta},\ m_k \in \mathbf{Z}. \tag{3.1.2}$$

Elements in Δ are called **(real) roots**. For $k \in J$, put $\alpha_k := \delta_k - \theta$. Then $\alpha_k \in \Delta$. For each $\mu \in \Delta$, let r_μ be the reflection in μ with respect to $(\cdot\,|\,\cdot)$ i.e.,

$$r_\mu\lambda := \lambda - (\lambda\,|\,\mu)\mu. \tag{3.1.3}$$

So $r_\mu \in \mathrm{Aut}_{\mathbf{Z}}(\Gamma)$. In particular, we write $r_i = r_{\alpha_i}$, $i \in I = \{1, \ldots, \ell+n\}$. For each $k \in J$, define

$$\Pi_k := \{\alpha_i\,|\,i \in I_k\}$$

$$\Delta_k := \dot{\Delta} + \mathbf{Z}\delta_k$$

$$Q_k := \bigoplus_{i\in I_k} \mathbf{Z}\alpha_i = \dot{Q} \oplus \mathbf{Z}\delta_k$$

$$\Gamma_k := Q_k \oplus \mathbf{Z}\Lambda_k$$

$$W_k := \langle r_i\,|\,i \in I_k\rangle < \mathrm{Aut}_{\mathbf{Z}}(\Gamma).$$

Also,

$$Q := Q^{[n]} := \sum_{i\in I} \mathbf{Z}\alpha_i = \dot{Q} \oplus \bigoplus_{k\in J} \mathbf{Z}\delta_k, \quad \text{called the } \textbf{n-toroidal root lattice,}$$

$$W := W^{[n]} := \langle r_i\,|\,i \in I\rangle < \mathrm{Aut}_{\mathbf{Z}}(\Gamma), \quad \text{called the } \textbf{n-toroidal Weyl group,}$$

$$\Pi := \Pi^{[n]} := \{\alpha_1, \ldots, \alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_{\ell+n}\}, \quad \text{called the set of } \textbf{simple roots.}$$

The following facts are clear.

LEMMA 3.1.1.

(i) $(\cdot\,|\,\cdot)$ *restricted to* $\Gamma \times \Gamma$, $\Gamma_k \times \Gamma_k$, $\dot{Q} \times \dot{Q}$, $\forall k \in J$ *are all symmetric nondegenerate and* W*-invariant.*

(ii) For each $k \in J$, *the restriction of* W_k *to* Γ_k *affords a faithful representation of the affine Weyl group* W_a *of type* $X_\ell^{(1)}$. *Hence* $W_k \cong W_a$. *Indeed* Δ_k, Q_k, W_k *can be viewed as an affine real root system, affine root lattice and affine Weyl group, respectively.*

(iii) The group generated by $\{r_i \,|\, i \in \dot{I}\}$ *is isomorphic to the finite Weyl group* $\dot{W}$. $\dot{W}$ *acts on* $\dot{Q}$ *faithfully.* □

LEMMA 3.1.2.

(i) $W\Delta = \Delta$ *and* $W\Pi = \Delta$.

(ii) For each $w \in W$, $\mu \in \Delta$, $wr_\mu w^{-1} = r_{w\mu}$. *Hence* $W = \langle r_\mu \,|\, \mu \in \Delta \rangle$.

PROOF: Since $\dot{W}$ leaves $\dot{\Delta}$ invariant and fixes δ_k, $k \in J$, $\dot{W}\Delta \subset \Delta$. It is enough to show that for $k \in J$, $r_k\Delta \subseteq \Delta$. Let μ be as in (3.1.2). Then

$$\begin{aligned} r_k\mu &= r_k\Big(\alpha + \sum_{j\in J} m_j\delta_j\Big) \\ &= \alpha + \sum_{j\in J} m_j\delta_j - \Big(\alpha + \sum_{j\in J} m_j\delta_j \,|\, \delta_k - \theta\Big)(\delta_k - \theta) \\ &\equiv \alpha - (\alpha\,|\,\theta)\theta \qquad \mod \sum_{j\in J} \mathbf{Z}\delta_j \\ &= r_\theta\alpha \in \dot{\Delta}. \end{aligned}$$

So $r_k\mu \in \Delta$. Now $W\Pi = \Delta$ follows from $W_k\Pi_k = \Delta_k$ and $W\delta_k = \delta_k$ which is standard. □

For $j \in J$ and $\alpha \in Q$, we define the linear maps on Γ by

$$t_\alpha^{(j)}\lambda = \lambda + (\lambda\,|\,\delta_j)\alpha - \Big[(\lambda\,|\,\alpha) + \frac{1}{2}(\alpha\,|\,\alpha)(\lambda\,|\,\delta_j)\Big]\,\delta_j \qquad (3.1.4)$$

Note that for $\beta \in Q$,

$$t_\alpha^{(j)}(\beta) = \beta - (\beta \,|\, \alpha)\delta_j. \tag{3.1.5}$$

LEMMA 3.1.3. *For each $j \in J$, we have*

(i) $t_\alpha^{(j)} t_\beta^{(j)} = t_{\alpha+\beta}^{(j)}$ for $\alpha, \beta \in Q$. Hence $t_\alpha^{(j)} \in Aut(\Gamma)$.

(ii) For any $w \in W$, $\alpha \in Q$ $w\, t_\alpha^{(j)}\, w^{-1} = t_{w\alpha}^{(j)}$.

(iii) $r_j = t_\theta^{(j)} \cdot r_\theta$.

PROOF: We simply write $t_\alpha = t_\alpha^{(j)}$ in this proof. For $\alpha, \beta \in Q$, $\lambda \in \Gamma$

$$\begin{aligned}
t_\alpha t_\beta \lambda &= t_\alpha \left(\lambda + (\lambda \,|\, \delta_j)\beta - \left[(\lambda \,|\, \beta) + \frac{1}{2}(\beta \,|\, \beta)(\lambda \,|\, \delta_j) \right] \delta_j \right) \\
&= t_\alpha \lambda + (\lambda \,|\, \delta_j) t_\alpha \beta - \left[(\lambda \,|\, \beta) + \frac{1}{2}(\beta \,|\, \beta)(\lambda \,|\, \delta_j) \right] t_\alpha \delta_j \\
&= \lambda + (\lambda \,|\, \delta_j)\alpha - \left[(\lambda \,|\, \alpha) + \frac{1}{2}(\alpha \,|\, \alpha)(\lambda \,|\, \delta_j) \right] \delta_j \\
&\qquad + (\lambda \,|\, \delta_j)(\beta - (\beta \,|\, \alpha)\delta_j) - \left[(\lambda \,|\, \beta) + \frac{1}{2}(\beta \,|\, \beta)(\lambda \,|\, \delta_j) \right] \delta_j \\
&= \lambda + (\lambda \,|\, \delta_j)(\alpha + \beta) - \left[(\lambda \,|\, \alpha + \beta) + \frac{1}{2}(\alpha + \beta \,|\, \alpha + \beta)(\lambda \,|\, \delta_j) \right] \delta_j \\
&= t_{\alpha+\beta}\, \lambda.
\end{aligned}$$

Therefore, $t_\alpha\, t_\beta = t_{\alpha+\beta}$. This gives (i).

For (ii) we note that $(\cdot \,|\, \cdot)$ is W-invariant and $w\delta_j = \delta_j$ for any $w \in W$. Thus

$$\begin{aligned}
w t_\alpha w^{-1} \lambda &= w \left(w^{-1}\lambda + (w^{-1}\lambda \,|\, \delta_j)\alpha - \left[(w^{-1}\lambda \,|\, \alpha) + \frac{1}{2}(\alpha \,|\, \alpha)(w^{-1}\lambda \,|\, \delta_j) \right] \delta_j \right) \\
&= \lambda + (\lambda \,|\, \delta_j) w\alpha - \left[(\lambda \,|\, w\alpha) + \frac{1}{2}(w\alpha \,|\, w\alpha)(\lambda \,|\, \delta_j) \right] \delta_j \\
&= t_{w\alpha} \lambda,
\end{aligned}$$

and hence, $w t_\alpha w^{-1} = t_{w\alpha}$. Finally,

$$r_j r_\theta \lambda = r_j(\lambda - (\lambda \,|\, \theta)\,\theta)$$

$$= \lambda - (\lambda \mid \alpha_j)\alpha_j - (\lambda \mid \theta)(\theta - (\theta \mid \alpha_j)\alpha_j)$$

$$= \lambda - (\lambda \mid \delta_j - \theta)(\delta_j - \theta) - (\lambda \mid \theta)\Big[\theta - (\theta \mid \delta_j - \theta)(\delta_j - \theta)\Big]$$

$$= \lambda - \Big[-(\lambda \mid \delta_j - \theta) - (\lambda \mid \theta)\Big]\theta - \Big[(\lambda \mid \delta_j - \theta) - 2(\lambda \mid \theta)\Big]\delta_j$$

$$= \lambda + (\lambda \mid \delta_j)\theta - \left[(\lambda \mid \theta) + \frac{1}{2}(\theta \mid \theta)(\lambda \mid \delta_j)\right] \delta_j$$

$$= t_\theta \lambda.$$

Thus $r_j r_\theta = t_\theta$ and $r_j = t_\theta r_\theta$. □

For $j \in J$ define

$$T^{(j)} := \langle t_\alpha^{(j)} \mid \alpha \in Q \rangle < \mathrm{Aut}\,(\Gamma)$$

$$\dot{H}_j := \langle t_\alpha^{(j)} \mid \alpha \in \dot{Q} \rangle, \qquad H := \langle t_\alpha^{(j)} \mid \alpha \in Q,\ j \in J \rangle.$$

LEMMA 3.1.4. *For each j, $T^{(j)} \cong Q/\mathbb{Z}\delta_j$. In particular, $T^{(j)}$ is abelian.*

PROOF: $\alpha \longmapsto t_\alpha^{(j)}$ gives a surjective homomorphism because of Lemma 3.1.3(i). Suppose that $t_\alpha^{(j)} = 1$. Then $t_\alpha^{(j)}(\Lambda_j) = \Lambda_j$ implies

$$(\Lambda_j \mid \delta_j)\alpha - \left[(\Lambda_j \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha)(\Lambda_j \mid \delta_j)\right] \delta_j = 0,$$

$$\alpha - \left[(\Lambda_j \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha)\right] \delta_j = 0.$$

It follows that $\alpha \in \mathbb{Z}\delta_j$. On the other hand, $t_{k\delta_j}^{(j)} = 1$ for all $k \in \mathbb{Z}$. Thus

$$0 \to \mathbb{Z}\delta_j \hookrightarrow Q \to T^{(j)} \to 1$$

is exact and $Q/\mathbb{Z}\delta_j \cong T^{(j)}$. □

LEMMA 3.1.5.

(i) For all $j \in J$, $T^{(j)} \lhd W$.

(ii) $H \triangleleft W$.

PROOF: In view of Lemma 3.1.3(ii) it will suffice to show that for each $\alpha \in Q$, $j \in J$, $t_\alpha^{(j)} \in W$. Since $t_\alpha^{(j)} t_\beta^{(j)} = t_{\alpha+\beta}^{(j)}$, it suffices to show $t_{\alpha_i}^{(j)} \in W$, $\alpha_i \in \Pi$. By Lemma 3.1.3(iii), we know $t_\theta^{(j)} = r_j r_\theta \in W$. Since $\dot{W}$ is transitive on $\dot{\Delta}$ there is a $w \in \dot{W}$ such that $w\theta = \alpha_i$. Finally by Lemma 3.1.3(ii), $t_{\alpha_i}^{(j)} = w\, t_\theta^{(j)}\, w^{-1} \in W$. □

Let $\mathfrak{t}_{ij} := t_{\delta_j}^{(i)}$ for $i, j \in J$. We note that

$$\mathfrak{t}_{ij}\lambda = \lambda + (\lambda \,|\, \delta_i)\delta_j - (\lambda \,|\, \delta_j)\delta_i, \qquad \mathfrak{t}_{ii} = 1, \quad \text{and} \quad \mathfrak{t}_{ij}\mathfrak{t}_{ji} = 1. \tag{3.1.6}$$

LEMMA 3.1.6. *The subgroup* $Z := \langle \mathfrak{t}_{ij} \,|\, i, j \in J \rangle$ *of* H *is a free abelian group on generators* $\{\mathfrak{t}_{ij} \,|\, i, j \in J,\ i < j\}$ *of rank* $\dfrac{n(n-1)}{2}$. *Moreover,* $Z \subseteq Cent(H)$.

PROOF: Note that $t_\alpha^{(j)}(\delta_k) = \delta_k$, for all $\alpha \in Q$, $j, k \in J$. Then $t_\alpha^{(j)}\, \mathfrak{t}_{pq} = \mathfrak{t}_{pq}\, t_\alpha^{(j)}$ follows from the W-invariance of the form $(\cdot\,|\,\cdot)$ and $W\delta_k = \delta_k$. Then Z is abelian and contained in the center of H. Since $\mathfrak{t}_{ij}\mathfrak{t}_{ji} = 1$, $\mathfrak{t}_{ii} = 1$, Z is generated by $\{\mathfrak{t}_{ij} \,|\, i < j\}$. Suppose that

$$\prod_{i<j} \mathfrak{t}_{ij}^{n_{ij}} = 1.$$

Then for all $\lambda \in \Gamma$,

$$\lambda = \prod_{i<j} \mathfrak{t}_{ij}^{n_{ij}} \lambda = \lambda + \sum_{i<j} n_{ij}\big((\lambda \,|\, \delta_i)\delta_j - (\lambda \,|\, \delta_j)\delta_i\big)$$

and hence

$$\sum_{i<j} n_{ij}\big((\lambda \,|\, \delta_i)\delta_j - (\lambda \,|\, \delta_j)\delta_i\big) = 0, \qquad \text{for all } \lambda \in \Gamma.$$

Substituting $\lambda = \Lambda_k$, $k \in J$, we find that $n_{ij} = 0$ for all $i < j$, $i, j \in J$. So Z is free. □

LEMMA 3.1.7. *Let* $\alpha, \beta \in Q$. *Then for all* $i, j \in J$

$$(t_\alpha^{(i)}, t_\beta^{(j)}) = t_\alpha^{(i)}\, t_\beta^{(j)}\, t_{-\alpha}^{(i)}\, t_{-\beta}^{(j)} = \mathfrak{t}_{ij}^{(\alpha\,|\,\beta)},$$

where $(x,y) = xyx^{-1}y^{-1}$ *is the group commutator.*

PROOF:

$$t_\alpha^{(i)} t_\beta^{(j)} \lambda = t_\alpha^{(i)} \left(\lambda + (\lambda \,|\, \delta_j)\beta - \left[(\lambda \,|\, \beta) + \frac{1}{2}(\beta \,|\, \beta)(\lambda \,|\, \delta_j) \right] \delta_j \right)$$

$$= \lambda + (\lambda \,|\, \delta_i)\alpha - \left[(\lambda \,|\, \alpha) + \frac{1}{2}(\alpha \,|\, \alpha)(\lambda \,|\, \delta_i) \right] \delta_i$$

$$+ (\lambda \,|\, \delta_j)[\beta - (\beta \,|\, \alpha)\delta_i] - \left[(\lambda \,|\, \beta) + \frac{1}{2}(\beta \,|\, \beta)(\lambda \,|\, \delta_j) \right] \delta_j$$

and

$$t_\beta^{(j)} t_\alpha^{(i)} \lambda = \lambda + (\lambda \,|\, \delta_j)\beta - \left[(\lambda \,|\, \beta) + \frac{1}{2}(\beta \,|\, \beta)(\lambda \,|\, \delta_j) \right] \delta_j$$

$$+ (\lambda \,|\, \delta_i)[\alpha - (\alpha \,|\, \beta)\delta_j] - \left[(\lambda \,|\, \alpha) + \frac{1}{2}(\alpha \,|\, \alpha)(\lambda \,|\, \delta_i)\delta_i \right].$$

Thus

$$t_\alpha^{(i)} t_\beta^{(j)} \lambda = t_\beta^{(j)} t_\alpha^{(i)} \lambda + (\alpha \,|\, \beta)[(\lambda \,|\, \delta_i)\delta_j - (\lambda \,|\, \delta_j)\delta_i]$$

$$= t_\beta^{(j)} t_\alpha^{(i)} \lambda + (\alpha \,|\, \beta)[(t_\beta^{(j)} t_\alpha^{(i)} \lambda \,|\, \delta_i)\delta_j - (t_\beta^{(j)} t_\alpha^{(i)} \lambda \,|\, \delta_j)\delta_i]$$

$$= \phi_{ij}^{(\alpha|\beta)} t_\beta^{(j)} t_\alpha^{(i)} \lambda.$$

Finally, $t_\alpha^{(i)} t_\beta^{(j)} t_{-\alpha}^{(i)} t_{-\beta}^{(j)} = \phi_{ij}^{(\alpha|\beta)}$. □

PROPOSITION 3.1.8. $H = (\prod_{i \in J} H_i)Z$ *(independent of the order of the product).*

PROOF: Since $\phi_{ij} = T_{\delta_j}^{(i)}$, we see from Lemma 3.1.3(i) that $\{t_\alpha^{(i)} \,|\, \alpha \in \dot{Q},\ i \in J\} \cup \{c_{ij} \,|\, i, j \in J\}$ generates H. Now Lemma 3.1.7 together with $Z \subseteq Cent(H)$ shows that any element in H can be expressed as

$$z \cdot \prod_{i \in J} t_{\beta_i}^{(i)} \tag{3.1.7}$$

for $\beta_i \in \dot{Q}$, $z \in Z$, the order of the product being arbitrary. □

COROLLARY 3.1.9. *If $h \in H$ has an expression of form (3.1.7),*

$$h = z \prod_{i \in J} t_{\beta_i}^{(i)}.$$

Then β_i is independent of the choice of the order of the product.

PROOF: From (3.1.4) and (3.1.5), we see that

$$t_\alpha^{(j)} \Lambda_k \equiv \Lambda_k + \delta_{jk}\alpha \qquad \text{mod } \textstyle\sum_{i \in J} \mathbf{Z}\delta_i,$$

$$t_\alpha^{(j)} \beta \equiv \beta \qquad \text{mod } \textstyle\sum_{i \in J} \mathbf{Z}\delta_i.$$

Thus

$$h\Lambda_k \equiv z t_{\beta_k}^{(k)} \Lambda_k \equiv \Lambda_k + \beta_k \qquad \text{mod } \textstyle\sum_{i \in J} \mathbf{Z}\delta_i.$$

□

COROLLARY 3.1.10.

(i) $Z = Cent(H)$.

(ii) $(H, H) = Z$.

(iii) H/Z is isomorphic to the direct product of n copies of $\dot{Q}$.

PROOF: Let $h = z \prod_{i \in J} t_{\beta_i}^{(i)}$ be in the form (3.1.7). Then $(h, t_\alpha^{(k)}) = 1$ for all $\alpha \in \dot{Q}$, all $k \in J$, implies that

$$1 = \left(\prod_{i \in J} t_{\beta_i}^{(i)}, t_\alpha^{(k)} \right) = \prod_{i \in J} (t_{\beta_i}^{(i)}, t_\alpha^{(k)}) = \prod_{i \in J} t_{ik}^{(\beta_i \,|\, \alpha)}.$$

Then by Lemma 3.1.6, $(\beta_i \,|\, \alpha) = 0$ for all $i \in J$, all $\alpha \in \dot{Q}$. So $\beta_i = 0$ for all $i \in J$ and $h = z \in Z$. $(H, H) = Z$ follows again from Lemma 3.1.7. □

PROPOSITION 3.1.11.

(i) $W = \dot{W} \ltimes H$.

(ii) $Z = Cent(W)$.

PROOF: Since $r_j = t_\theta^{(j)} r_\theta$ $(j \in J)$ (Lemma 3.1.3(iii)), $W = \langle \dot{W}, H \rangle$. We also have $H \lhd W$ (Lemma 3.1.5). It remains to check $\dot{W} \cap H = \{1\}$. Let $h = z \prod_{j \in J} t_{\beta_j}^{(j)}$ be of the form (3.1.7) and suppose that it belongs to $\dot{W}$. Then $h\Lambda_k = \Lambda_k$. Also as in the proof of Corollary 3.1.9, we have

$$h\Lambda_k = z \prod_{j \in J} t_{\beta_j}^{(i)} = \Lambda_k + \beta_k \qquad (\mod \textstyle\sum_{i \in J} \mathbf{Z}\delta_i).$$

So $\beta_k = 0$ for all $k \in J$, and $h = z \in Z \cap \dot{W}$. Since any element in Z acts on Q as 1 and any element in $\dot{W}$ act on $\sum_{k \in J} \mathbf{Z}\Lambda_k$ as 1, we see that $h = 1$. □

REMARKS: (i) One also may deduce $\dot{W} \cap H = 1$ by proving that H is a torsion free group by using (3.1.7) and Corollary 3.1.9.

(ii) According to Corollary 3.1.10, the group H is a Heisenberg group (cf. Appendix).

Up to now, we have worked for a fixed integer n. Now we let it vary. Then consider the natural inclusions:

$$\Gamma^{[1]} \subset \Gamma^{[2]} \subset \cdots \subset \Gamma^{[n]}$$

$$Q^{[1]} \subset Q^{[2]} \subset \cdots \subset Q^{[n]}$$

$$W^{[1]} \subset W^{[2]} \subset \cdots \subset W^{[n]}.$$

Since $\Gamma^{[k]}$ affords a faithful representation for $W^{[k]}$ (Lemma 3.1.1(ii)), we may view $W^{[k]}$ either as a subgroup of $W^{[n]}$ $(k \leq n)$ or the group defined as before directly from the lattice $\Gamma^{[k]}$.

We want to prove

PROPOSITION 3.1.12. $W^{[n]} \cong W^{[n-1]} \ltimes Q^{[n-1]}$.

The proposition says that the n-toroidal Weyl group can be constructed from $(n-1)$-toroidal Weyl group and its root lattice.

PROOF: By Lemma 3.1.4 we obtain an isomorphism

$$Q^{[n-1]} \cong Q^{[n]}/\mathbb{Z}\delta_{\ell+n} \cong T^{(\ell+n)}$$

$$\alpha \longmapsto t_\alpha^{(\ell+n)}.$$

It suffices to prove that $W^{[n]} \cong W^{[n-1]} \ltimes T^{(\ell+n)}$ and the action of $W^{[n]}$ on $T^{(\ell+n)}$ is compatible with that on $Q^{[n-1]}$ by the above isomorphism. The latter fact is from Lemma 3.1.3(ii). Also by Lemma 3.1.3(i), $r_{\ell+n} = t_\theta^{(\ell+n)} r_\theta \in T^{(\ell+n)} \cdot \dot{W}$ from which we have $W^{[n]} = W^{[n-1]} T^{(\ell+n)}$.

Suppose that $w = t_\alpha^{(n)} \in W^{[n-1]} \cap T^{(\ell+n)}$ for $\alpha \in Q^{[n-1]}$. Then

$$\Lambda_n = w\Lambda_n = t_\alpha^{(n)}\Lambda_n = \Lambda_n + (\Lambda_n \,|\, \delta_n)\alpha - \left[(\Lambda_n \,|\, \alpha) + \frac{1}{2}(\alpha \,|\, \alpha)(\Lambda_n \,|\, \delta_n)\right] \delta_n$$

$$= \Lambda_n + \alpha - \frac{1}{2}(\alpha \,|\, \alpha)\delta_n.$$

So $\alpha = 0$, i.e., $w = t_0^{(n)} = 1$. □

COROLLARY 3.1.13. $W^{[n]} = \Big(\Big(\big((\dot{W} \ltimes \dot{Q}) \ltimes Q^{[1]}\big) \ltimes Q^{[2]}\Big) \ltimes \cdots\Big) \ltimes Q^{[n-1]}$.

□

§3.2. Relations to some indefinite Kac-Moody Weyl groups.

In this section, we emphasize on relations between the n-toroidal Weyl groups, for a *fixed positive integer* n and some indefinite type Kac-Moody Weyl groups. We keep the notation in §3.1.

Recall the set of simple roots of Δ is

$$\Pi = \{\alpha_i \,|\, i \in I\}.$$

Let $A = A^{[n]}$ be the matrix $((\alpha_i \,|\, \alpha_j))_{i,j\in I}$. Then

$$A^{[n]} = \begin{bmatrix} \dot{A} & ((\alpha_i \,|\, \alpha_j))_{i\in \dot{I}, j\in J} \\ ((\alpha_i \,|\, \alpha_j))_{i\in \dot{I}, j\in J} & \begin{matrix} 2 & 2 & \dots & 2 \\ 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \dots & 2 \end{matrix} \end{bmatrix}$$

where $\dot{A}$ has the dimension $\ell \times \ell$ and the block of 2's has the dimension $n \times n$. This is a generalized intersection matrix of Slodowy [Sl].

By changing 2's off diagonal to -2, we get a generalized Cartan matrix $\hat{A} = \hat{A}^{[n]}$,

$$\hat{A} = \begin{bmatrix} \dot{A} & ((\alpha_i \,|\, \alpha_j))_{i\in \dot{I}, j\in J} \\ ((\alpha_i \,|\, \alpha_j))_{i\in \dot{I}, j\in J} & \begin{matrix} 2 & -2 & \dots & -2 \\ -2 & 2 & \dots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \dots & 2 \end{matrix} \end{bmatrix}.$$

Let $\hat{\Pi} := \{\hat{\alpha}_1, \dots, \hat{\alpha}_\ell, \hat{\alpha}_{\ell+1}, \dots, \hat{\alpha}_{\ell+n}\}$ be the set of simple roots of the real root system $\hat{\Delta}$ of $\hat{A}$ so that $\{\hat{\alpha}_1, \dots, \hat{\alpha}_\ell\}$ is a set of simple roots for the finite root system $\dot{\Delta}$. Identity $\alpha_i = \hat{\alpha}_i, \quad i \in \dot{I}$. One notices that for each $j \in J$, $\hat{\Pi}_j := \{\hat{\alpha}_i \,|\, i \in I_j\}$ forms a set of simple roots of an affine root system. Denote by $\hat{\delta}_j$ the null root relative to $\hat{\Pi}_j$, i.e., $\hat{\delta}_j = \theta + \hat{\alpha}_j$.

Convention: Objects with "$\hat{}$" are these relevant to the generalized Cartan matrix $\hat{A}$. Objects without "$\hat{}$" are these relevant to the generalized intersection matrix A. They denote similar notion, respectively. In some situations, they look the same after the identification $\alpha_i = \hat{\alpha}_i, \quad i \in \dot{I}$. e.g. θ denotes the highest root relative to either $\{\alpha_1, \dots, \alpha_\ell\}$ or $\{\hat{\alpha}_1, \dots, \hat{\alpha}_\ell\}$.

For completeness, we list the following:

$$\hat{Q} = \bigoplus_{i=1}^{\ell+n} \mathbf{Z}\hat{\alpha}_i = \bigoplus_{j=1}^{\ell} \mathbf{Z}\hat{\alpha}_i \oplus \bigoplus_{j=\ell+1}^{\ell+n} \mathbf{Z}\hat{\delta}_j, \quad \text{the } \textit{root lattice} \text{ of } \hat{A},$$

$$\hat{Q}_j = \dot{Q} + \mathbf{Z}\hat{\delta}_j, \qquad j \in J, \quad \text{the } \textit{affine root lattices}.$$

$\langle \cdot \,|\, \cdot \rangle$, the symmetric bilinear, nondegenerate form on $\hat{Q}$,

$\hat{r}_i = \hat{r}_{\hat{\alpha}_i}$, the simple reflections in α_i, $i \in I$,

$\hat{r}_\alpha$, the reflection in $\alpha \in \hat{\Delta}$,

$\hat{W}$, the indefinite Weyl group $= \langle \hat{r}_i \,|\, i \in I \rangle = \langle \hat{r}_\alpha \,|\, \alpha \in \hat{\Delta} \rangle$,

$\hat{W}_j$, $j \in J$, the subgroup of W generated by $r_1, \ldots, r_\ell, \hat{r}_j$.

The following facts about Weyl groups are well-known (see for instances, [MP] and [Ka]).

W1. $\hat{Q}$ is a faithful representation for $\hat{W}$ since $\langle \cdot | \cdot \rangle$ is nondegenerate on $\hat{Q}$.

W2. $\langle \hat{\delta}_i \,|\, \hat{\delta}_j \rangle = -4 \neq 0$, $i, j \in J$.

W3. $\hat{W}_j \cong W_j \cong \dot{W} \ltimes \dot{Q}$ *canonically.*

W4. $\hat{W}$ is a Coxeter group with the presentation

generators : $\hat{r}_i$, $i \in I$

relations : $\hat{r}_i = 1$, $i \in I$

$(\hat{r}_i \hat{r}_j)^2 = 1$ if $\langle \hat{\alpha}_i \,|\, \hat{\alpha}_j \rangle = 0$

$(\hat{r}_i \hat{r}_j)^3 = 1$ if $\langle \hat{\alpha}_i \,|\, \hat{\alpha}_j \rangle = -1$,

LEMMA 3.2.1. *For $i \neq j \in J$, $\hat{W}_i \cap \hat{W}_j = \dot{W}$.*

PROOF: Let $w \in \hat{W}_i \cap \hat{W}_j$. Defining $\hat{t}_\alpha^{(i)}$ in the same way as $t_\alpha^{(i)}$ for $\alpha \in \dot{Q}$, we see that there exist $\alpha \in \dot{Q}$, $\beta \in \dot{Q}$ such that $\hat{t}_\alpha^{(i)} w$, $\hat{t}_\beta^{(j)} w \in \dot{W}$ (W3). We know that $\hat{W}_i \hat{\delta}_i = \hat{\delta}_i$ and $\hat{W}_j \hat{\delta}_j = \hat{\delta}_j$. Thus

$$\hat{\delta}_i = \hat{t}_\alpha^{(i)} w \hat{\delta}_i = \hat{t}_\beta^{(j)} w \hat{\delta}_i = \hat{t}_\beta^{(j)} \hat{\delta}_i$$

$$= \hat{\delta}_i + \langle \hat{\delta}_i \,|\, \hat{\delta}_j \rangle \beta - \left[\langle \beta \,|\, \hat{\delta}_i \rangle + \frac{1}{2} \langle \beta \,|\, \beta \rangle \langle \hat{\delta}_i \,|\, \hat{\delta}_j \rangle \right] \hat{\delta}_j.$$

This forces $\beta = 0$ since $(\hat{\delta}_i \,|\, \hat{\delta}_j) \neq 0$ (W2). So $w \in \dot{W}$. □

PROPOSITION 3.2.2. $\hat{W} \cong \hat{W}_{\ell+1} *_{\dot{W}} \hat{W}_{\ell+2} *_{\dot{W}} \cdots *_{\dot{W}} \hat{W}_{\ell+n}$, *i.e.,* $\hat{W}$ *is the free product of* $\hat{W}_j$ $(j \in J)$ *amalgamated over* $\dot{W}$.

PROOF: By Lemma 3.2.1 the amalgamated product

$$F := \hat{W}_{\ell+1} *_{\dot{W}} \hat{W}_{\ell+2} *_{\dot{W}} \cdots *_{\dot{W}} \hat{W}_{\ell+n}$$

exists. The inclusions $\hat{W}_j \hookrightarrow \hat{W}$ lead to a surjective homomorphism $f : F \to \hat{W}$. On the other hand, $\hat{W}_{\ell+k}$ also can be viewed as a subgroup of F. Moreover $\hat{r}_i, i \in I$ generate the group F by the structure of free products. Since each of relations in W4 is from only one of affine Weyl groups $\hat{W}_k$, $k \in J$. Then by W4, $\hat{r}_i \to \tilde{r}_i$ defines a homomorphism from $\hat{W} \to F$. This is precisely the inverse map of f and we conclude

$$\hat{W} \cong \hat{W}_{\ell+1} *_{\dot{W}} \cdots *_{\dot{W}} \hat{W}_{\ell+n}.$$

□

Recall a characteristic property for a free product of groups:

$$G = G_1 *_{G_0} G_2 *_{G_0} \cdots *_{G_0} G_n.$$

If $\{1\} \cup \{g_{k_i}\}$ is a set of coset representatives for G_k/G_0, then every element $g \in G$ can be expressed uniquely in the form

$$g = g_0 x_1 \cdots x_m$$

where

- $m \in \mathbb{Z}_{\geq 0}$,
- $x_i \in \bigcup_k (\{g_{k_i}\}_i)$, $g_0 \in G_0$ (3.2.1)
- no two consecutive x_i's lie in the same group G_j.

Now we use this property to give another description of $\hat{W}$.

PROPOSITION 3.2.3. *$\hat{W} \cong \dot{W} \ltimes (\dot{Q} * \dot{Q} * \cdots * \dot{Q})$ where $\dot{Q} * \dot{Q} * \cdots * \dot{Q}$ is the free product of n copies $\dot{Q}$.*

PROOF: By the Proposition 3.2.2 $\hat{W} \cong \hat{W}_{\ell+1} *_{\dot{W}} \cdots *_{\dot{W}} \hat{W}_n$. Since $\hat{W}_j = \dot{W} \ltimes \dot{Q}$, where $\alpha \longleftrightarrow t_\alpha^{(j)} \longleftrightarrow \hat{t}_\alpha^{(j)}$, we naturally take representatives

$$\hat{t}_\alpha^{(j)}, \quad \alpha \in \dot{Q} \qquad \text{for} \quad \hat{W}_j/\dot{W}.$$

Then by (3.2.1) any element $\hat{w}$ in $\hat{W}$ has a unique expression

$$\hat{w} = w x_1 \cdots x_m \qquad \text{with} \quad w \in \dot{W}, \tag{3.2.2}$$

$$x_k \in \{\hat{t}_\alpha^{(j)},\ \alpha \in \dot{Q} \setminus \{0\},\ j \in J\},$$

and

$$\{x_k, x_{k+1}\} \not\subset \{\hat{t}_\alpha^{(j)} \mid \alpha \in \dot{Q}\}, \qquad k = 1, \ldots, m.$$

for all $j \in J$. But elements in $\dot{W} \ltimes (\dot{Q} * \dot{Q} * \cdots * \dot{Q})$ also have the same unique expression and we easily obtain

$$\hat{W} \cong \hat{W}_{\ell+1} *_{\dot{W}} \cdots *_{\dot{W}} \hat{W}_{\ell+n} \cong \dot{W} \ltimes (\dot{Q} * \dot{Q} * \cdots * \dot{Q}).$$

□

Now we return to toroidal Weyl groups. The toroidal Weyl group W has generators $\{r_i \mid i \in I\}$.

LEMMA 3.2.4. *$\{r_i \mid i \in I\}$ satisfies the relations stated in W4 (replacing $\hat{r}_i$ by r_i).*

PROOF: This is clear since any pair (r_i, r_j) appearing in these relations sits in one of affine Weyl groups. □

From this lemma and W4, we see that the map $\hat{r}_i \to r_i$ defines a surjective homomorphism, denoted by π, from the indefinite Kac-Moody Weyl group $\hat{W}$ onto W. In the rest of this section, we will find out the kernel of π.

Since $\hat{W} = \dot{W} \ltimes (\dot{Q} * \dot{Q} * \cdots * \dot{Q})$ (Proposition 3.2.3), then any element $\hat{w} \in \hat{W}$ has the unique expression of form (3.2.2), i.e.,

$$\hat{w} = w\, \hat{t}_{\beta_1}^{(i_1)}\, \hat{t}_{\beta_2}^{(i_2)} \cdots \hat{t}_{\beta_m}^{(i_m)}$$

where $w \in \dot{W}$, $\beta_k \in \dot{Q}$, $i_k \neq i_{k+1} \in J$. Then

$$\pi(\hat{w}) = w\, t_{\beta_1}^{(i_1)}\, t_{\beta_2}^{(i_2)} \cdots t_{\beta_m}^{(i_m)}.$$

Consider the map $\hat{\pi} : \hat{W} \to W/Z$ defined by the composition

$$\hat{W} \xrightarrow{\pi} W \xrightarrow{\text{nat}} W/Z \cong \dot{W} \ltimes H/Z.$$

Now suppose $\hat{w} \in \ker \pi$. Then in W/Z, $\hat{\pi}\,\hat{w} = w\, t_{\beta_1}^{(i_1)} \cdots t_{\beta_m}^{(i_m)} = 1$, and so using Corollary 3.1.9 and Corollary 3.1.10(iii), $w = 1$ and for all $k \in J$, $\sum_{i_j = k} \beta_{i_j} = 0$. So

$$\ker \pi \subset K := \left\{ \hat{t}_{\beta_1}^{(i_1)}\, \hat{t}_{\beta_2}^{(i_2)} \cdots \hat{t}_{\beta_m}^{(i_m)} \mid \sum_{i_j = k} \beta_{i_j} = 0 \quad \text{for all} \quad k \in J \right\}.$$

Let p_{ij} $(i < j,\ i, j \in J)$ be the $\mathbf{Z}$-valued projection map from Z onto $\langle \mathfrak{k}_{ij} \rangle \cong \mathbf{Z}$ $(\mathfrak{k}_{ij} \mapsto 1)$. We note that $\pi(K) \subseteq Z$. Then $\bar{p}_{ij} := p_{ij}\pi$ is a well defined map from K into $\mathbf{Z}$. So

$$\hat{h} = \hat{t}_{\beta_1}^{(i_1)}\, \hat{t}_{\beta_2}^{(i_2)} \cdots \hat{t}_{\beta_m}^{(i_m)} \in \ker \pi \cap K \iff \bar{p}_{ij}(\hat{h}) = 0 \qquad \text{for all} \quad i, j \in J,\ i < j.$$

So we have

PROPOSITION 3.2.5. *The map $\hat{r}_i \mapsto r_i$ defines a surjective homomorphism π from the indefinite Kac-Moody Weyl group $\hat{W}$ onto toroidal Weyl group W. Moreover,*

$$\ker \pi = \Big\{ \hat{h} = \hat{t}_{\beta_1}^{(i_1)}\, \hat{t}_{\beta_2}^{(i_2)} \cdots \hat{t}_{\beta_m}^{(i_m)} \mid \beta_i \in \dot{Q},\ 1 \leq i \leq m;$$

$$\sum_{i_j = k} \beta_j = 0 \quad \text{for all } k \in J;$$

$$\bar{p}_{ij}(\hat{h}) = 0 \quad \text{for all } i, j \in J \text{ and } i < j \Big\}$$

□

As one sees that $\overline{p}_{ij}(\hat{h})$ is calculable from Lemma 3.1.7.

§3.3. An affine variation.

Let Γ be any lattice in some real space $\mathbf{R}^N$ on which there is a non-degenerate symmetric bilinear form $(\cdot \mid \cdot)$ for which $(\alpha \mid \alpha) \in 2\mathbf{Z}$ for all $\alpha \in \Gamma$. Let S be a subset of $\{\alpha \in \Gamma \mid (\alpha \mid \alpha) = 2\}$ which is the set of elements in Γ with square length 2. In [EMY] Eswara Rao, Moody and Yokonuma studied the Lie algebras generated by the set of vertex operators $X(\alpha, z)$, $\alpha \in S$. In particular, in this theory there arises a "reflection" group $W(S, \mathbf{Z})$ generated by the involutions $r_\alpha^{(m)} : \Gamma \to \Gamma$ defined by

$$\lambda \mapsto \lambda - (\lambda \mid \alpha)\alpha - m\alpha \tag{3.3.1}$$

where $\alpha \in S, m \in \mathbf{Z}$.

A case of special importance is that when $\Gamma = \Gamma^{[n]}$ as defined in §3.1 and α runs through the elements of $\Delta = \dot{\Delta} + \mathbf{Z}\delta_{\ell+1} + \mathbf{Z}\delta_{\ell+2} + \cdots + \mathbf{Z}\delta_{\ell+n}$. The main result of this section is

PROPOSITION 3.3.1. $W(\Delta, \mathbf{Z}) \simeq W^{[n+1]}$.

The isomorphism will be made explicit in (3.3.2) and this propostion is a consequence of Proposition 3.3.2 below.

Let $\Gamma \subset \mathbf{R}^N$ be arbitrary even integral lattice and let $S \subset \Gamma$ be a nonempty set of vectors of square length 2. Let $W(S)$ be the group generated by the reflections

$$r_\alpha : \lambda \mapsto \lambda - (\lambda \mid \alpha)\alpha, \qquad \alpha \in S.$$

Let

$$\widehat{\Gamma} = \Gamma \perp (\mathbf{Z}\delta + \mathbf{Z}\Lambda) \subset \mathbf{R}^{N+2}$$

be an extension of $\mathbf{R}^N$ by a hyperbolic plane where $(\delta \mid \delta) = (\Lambda \mid \Lambda) = 0$, and $(\delta \mid \Lambda) = 1$ and let $\widehat{S} = S + \mathbf{Z}\delta$. We prove

PROPOSITION 3.3.2. $W(S, \mathbf{Z}) \simeq W(\widehat{S})$.

Clearly, Proposition 3.3.1 is an immediate consequence of this.

PROOF OF PROPOSITION 3.3.2: We consider the "hyperplane" of cosets

$$\Gamma_{aff} := (\Lambda + \Gamma + \mathbf{Z}\delta)/\mathbf{Z}\delta \subset \widehat{\Gamma}/\mathbf{Z}\delta.$$

Then we have a canonical bijection $\tau : \Gamma \to \Gamma_{aff}$ given by

$$\gamma \mapsto \Lambda + \gamma + \mathbf{Z}\delta$$

and a simply transitive action of Γ on Γ_{aff} defined by

$$+ : \ (\lambda, \Lambda + \gamma + \mathbf{Z}\delta) \mapsto \Lambda + (\gamma + \lambda) + \mathbf{Z}\delta.$$

Using the latter we view Γ_{aff} as an affine space with translation group Γ. Thus for $p = \tau(\lambda)$, $q = \tau(\mu)$, $p - q$ is by definition the unique element of Γ_{aff} such that $(p - q) + q = p$. Using τ we transfer the quadratic form $\|\lambda\|^2 := (\lambda \mid \lambda), \lambda \in \Gamma$, onto Γ_{aff}:

$$\|p - q\|^2 := \|\lambda - \mu\|^2.$$

Let $\hat{\beta} = \beta + k\delta \in \widehat{S}$ where $\beta \in S$. Since

$$\Lambda + \Gamma + \mathbf{Z}\delta = \{\mu \in \widehat{\Gamma} \mid (\mu \mid \delta) = 1\}$$

and since $r_{\hat{\beta}}\delta = \delta$, we see that $r_{\hat{\beta}}$ induces a transformation on Γ_{aff}. In fact, by (3.3.1),

$$\begin{aligned} r_{\hat{\beta}}\tau(\lambda) &= r_{\hat{\beta}}(\Lambda + \lambda + \mathbf{Z}\delta) \qquad (3.3.2) \\ &= (\Lambda + \lambda) - (\Lambda + \lambda \mid \beta + k\delta)(\beta + k\delta) + \mathbf{Z}\delta \\ &= \Lambda + r_\beta\lambda - k\beta + \mathbf{Z}\delta \\ &= \tau(r_\beta^{(k)}(\lambda)). \end{aligned}$$

This shows explicitely that there is a surjective homomorphism

$$\psi : W(\widehat{S}) \longrightarrow W(S, \mathbf{Z})$$

for which τ is covariant map. Since $W(\widehat{S})$ acts faithfully on Γ and τ is bijective, the map is an isomorphism. □

We note that obviously $W(S, \mathbf{Z}) \simeq W(S) \ltimes T$ where $T \simeq \sum_{\beta \in S} \mathbf{Z}\beta \subset \Gamma$ and $W(S)$ acts on T by restriction from Γ to T. In the case of Proposition 3.3.1, this shows again that $W^{[n+1]} \simeq W^{[n]} \cong Q^{[n]}$ (see Proposition 3.1.12).

Appendix

Deflnition of Heisenberg group of n copies of an abelian group with a form $(\cdot|\cdot)$.

Let A be a multiplicative abelian group with a *bimultiplicative* form $(\cdot|\cdot) : A \times A \to \mathbf{Z}$ such that $(\cdot|\cdot)$ is

$$\text{symmetric}: \quad (x|y) = (y|x)$$

$$\text{bimultiplicative}: \quad (xy|z) = (x|z) + (y|z)$$

$$(x|yz) = (x|y) + (x|z)$$

$$\text{nondegenerate}: \quad (x|y) = 0, \quad \text{for all } y \in A, \text{ implies } x = 1.$$

For a positive integer n, we take n copies A. To distinguish each copy, we let $A_i = A$. Then element in A_i will be specified as $a(i)$ for $a \in A$.

Let Z be the abstract abelian group with

$$\text{generators}: \quad z_{ij}, \qquad i, j = 1, \ldots, n$$

$$\text{relations}: \quad z_{ii} = 1, \; z_{ij} z_{ji} = 1, \quad 1 \le i, j \le n.$$

So Z is the free abelian group on $\dfrac{n(n-1)}{2}$ generators $\{z_{ij} | i < j\}$. Form the free product

$$\left(\overset{n}{\underset{i=1}{*}} A_i\right) * Z$$

from which we factor out the normal subgroup generated by the following relations

$$(\alpha(i), \beta(j)) = z_{ij}^{(\alpha|\beta)}$$

$$(z_{ij}, \alpha(k)) = 1$$

We call the quotient group the *Heisenberg group* of n copies A, denoted by $H^{[n]} = H^{[n]}(A)$, or simply H. A typical element in $H^{[n]}$ is

$$z \cdot \prod_{i=1}^{n} a_i(i), \qquad z \in Z,\ a_i \in A$$

and for a fixed order the expression is unique.

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Conclusion

In this thesis, we study certain groups involved in Lie theory. Namely, these are groups graded by finite root systems, toroidal groups and toroidal Weyl groups.

The common thread is the notion of Steinberg groups, associated with a finite root system and a ring, is an abstract group with certain generators and (Steinberg) relations. In Chpater 1, we introduce groups graded by finite root systems of type $A_\ell, \ell \geq 3, D_\ell, \ell \geq 4$, and $E_\ell, \ell = 6, 7, 8$, based on some properties of Steinberg groups. A ring can be constructed from such a group and then this group has Steinberg relations. Also we show that any covering of a group graded by a root system of rank at least 4 for A_ℓ and 5 for D_ℓ is also graded by this root system. In Chapter 2, by using integral bases of toroidal Lie algebras relative to simply laced finite root systems $\dot{\Delta}$, we construct toroidal groups, which generalizes affine Kac-Moody groups. Toroidal groups are homomorphic images of the Steinberg groups $St(\dot{\Delta}, \mathbb{C}[s^{\pm 1}, t^{\pm 1}])$. We study the toroidal groups via the vertex representations and show that in the case of $A_\ell, \ell \geq 4$, $St(\dot{\Delta}, \mathbb{C}[s^{\pm 1}, t^{\pm 1}])$ is a universal central extension of the toroidal group with the kernel $K_2(\mathbb{C})$. Also we show that finite Weyl groups, affine Weyl groups and toroidal Weyl groups are subquotient of the Steinberg groups. In Chpater 3, we discuss toroidal Weyl groups. It turns out a toroidal Weyl group is the semidirect product of the relative finite Weyl group and a discrete Heisenberg group.